NEW SECTIONS FOR HANDBOOK OF DIFFERENTIAL EQUATIONS

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Note: Ignore the "(Not cited.)" statement in each reference.

EXACT METHODS

1. Complexification

Applicable to Systems of first order ODEs in two variables whose forcing functions satisfy the Cauchy–Riemann equations.

Yields

Sometimes, an explicit solution.

Idea

Two ODEs may represent the real and complex parts of a single ODE when written in terms of a complex variable. When this occurs, and the single variable ODE can be solved, then the original system can be solved.

Procedure

Suppose we have the two ODEs

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x,y), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = g(x,y), \tag{1.1}$$

and suppose that the Cauchy–Riemann equations

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial y}, \qquad \frac{\partial f}{\partial y} = -\frac{\partial g}{\partial x}, \tag{1.2}$$

are satisfied. Then the ODEs in (1.1) can be written in terms of the single complex unknown $z = x + iy$ which satisfies the first order ODE $\frac{\mathrm{d}z}{\mathrm{d}t} = h(z)$ for some function h. This can be integrated as $\int^z \frac{\mathrm{d}w}{h(w)} = t$. Then $x(t)$ and $y(t)$ are the real and imaginary parts of $z(t)$.

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Example 1

First order system: The most general set of first order (i.e., linear) constant coefficient ODEs in two variables whose forcing functions satisfy the Cauchy–Riemann equations are

$$\begin{aligned} \dot{x} &= a + bx + cy, \\ \dot{y} &= A - cx + by, \end{aligned} \tag{1.3}$$

where $\{a, b, c, A\}$ are constants. For these equations the differential equation for $z = x + iy$ is

$$\dot{z} = (a + iA) + (b - ic)z, \qquad z_0 = x_0 + iy_0. \tag{1.4}$$

Its solution is $z(t) = e^{(b-ic)t}\left(z_0 + \dfrac{a+iA}{b-ic}\right) - \left(\dfrac{a+iA}{b-ic}\right)$. The solutions for $x(t)$ and $y(t)$ are the real and imaginary parts of $z(t)$.

Example 2

Second order system: The most general set of second order (i.e., having up to quadratic terms) constant coefficient ODEs in two variables whose forcing functions satisfy the Cauchy–Riemann equations are

$$\begin{aligned} \dot{x} &= a + bx + cy + dx^2 + 2exy - dy^2, \\ \dot{y} &= A - cx + by - ex^2 + 2dxy + ey^2. \end{aligned} \tag{1.5}$$

where $\{a, b, c, d, e, A\}$ are constants. For these equations the differential equation for $z = x + iy$ is

$$\dot{z} = (a + iA) + (b - ic)z + (d - ie)z^2, \qquad z_0 = x_0 + iy_0. \tag{1.6}$$

This can be solved explicitly: $z(t) = z_1 + \dfrac{z_2 - z_1}{1 - \left(\dfrac{z_0 - z_2}{z_0 - z_1}\right) e^{(d-ie)(z_2-z_1)t}}$ where $\{z_1, z_2\}$ are the roots of $(a + iA) + (b - ic)z + (d - ie)z^2 = 0$.

Example 3

Third order system: The system

$$\begin{aligned} \dot{x} &= -8 + 12x - 6x^2 + 6y^2 + x^3 - 3xy^2, && x(0) = 2, \\ \dot{y} &= 12y - 12xy + 3x^2y - y^3, && y(0) = 1, \end{aligned} \tag{1.7}$$

leads to the complex ODE $\dot{z} = (z - 2)^3$ with $z(0) = 2 + i$. Solving results in $z(t) = 2 + \dfrac{i}{\sqrt{1+2t}}$. Hence, the solutions of the original system are $x(t) = 2$, and $y(t) = \dfrac{1}{\sqrt{1+2t}}$.

Notes

1. This section was created from Kiprono and Tóth [1]. That paper also includes:
   (a) Explicit formulae, similar to (1.3) and (1.5), for third order equations (e.g., up to cubic terms) and for the general case.
   (b) Explicit formulae, similar to (1.3) and (1.4) for first order equations in four independent variables comprising two complex variables.
2. Complexifiation can solve some two-dimensional problems in linear elastostatics. For example, the Airy stress tensor ϕ (from which stress components can be found) in the plane (e.g., with axes x_1 and x_2) may, depending on the problem, satisfy the biharmonic equation

$$\nabla^4\phi = \frac{\partial^4\phi}{\partial {x_1}^4} + 2\frac{\partial^4\phi}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4\phi}{\partial {x_2}^4} = 0. \tag{1.8}$$

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Using the comples variable $z = x_1 + ix_2$, (1.8) results in $\nabla^4\phi = 16\dfrac{\partial^4\phi}{\partial z^2\partial\overline{z}^2} = 0$ whose most general solution is $\phi = f_1(z) + f_2(\overline{z}) + \overline{z}f_3(z) + zf_4(\overline{z})$ where the $\{f_i\}$ are general holomorphic functions; see Barber [2].

References

[1] Kelvin Kiprono and and János Tóth, *Symbolic solution of systems of polynomial differential equations via the Cauchy–Riemann equation. Applications to kinetic differential equations*, 2024. `https://arxiv.org/pdf/2402.09842` (Not cited.)

[2] J. R. Barber, *Linear Elastostatics*. `https://websites.umich.edu/~jbarber/UNESCO.pdf` (acessed 17 May 2025) (Not cited.)

2. Linear Operators Forced at Resonance

Applicable to Linear ODE operators, $L = a_0(x) + \sum_{j=1}^{n} a_j(x)\dfrac{\mathrm{d}^j}{\mathrm{d}x^j}$, forced at resonance, such as $L[y(x)] = f(x)$ where $L[f(x)] = 0$; this is equivalent to $L^2[y(x)] = 0$. It also applies to the more general case $L^m[y(x)] = 0$.

Yields

A representation of a particular solution.

Idea

By generalizing the L operator to be L_λ (dependent on a new parameter, λ), the solution to $L^m[y(x)] = 0$ can be written in terms of a solution $y(x;\lambda)$ to $L_\lambda[y(x;\lambda)] = 0$.

Procedure

Given the linear ODE operator L, choose a function $g(x)$ which is m times continuously differentiable. Then select a value λ_0 such that $g'(\lambda_0) \neq 0$. Define the generalized operator $L_\lambda = \widehat{L} - g(\lambda)$ such that $L_{\lambda_0} = L$; the examples will clarify the form of L_λ. (We do not explicitly use the $\widehat{L}$ operator, so we do not discuss it.)

Let $u(x)$ be a solution of the homogeneous problem for the L operator: $\mathrm{L}[y(x)] = 0$. Let $u(x;\lambda)$ be a solution of the homogeneous problem for the L_λ operator: $L_\lambda[y(x;\lambda)] = 0$. Then a particular solution to the resonantly forced equation $L_\lambda[y(x;\lambda)] = u(x;\lambda)$ is the function

$$\widehat{u}(x;\lambda) = \frac{1}{g'(\lambda)}\frac{\partial u(x;\lambda)}{\partial\lambda} \qquad \text{with } g'(\lambda) \neq 0 \tag{2.1}$$

Using this, a solution to the original resonantly forced problem, that is $L[y(x)] = u(x)$, is $y(x) = \widehat{u}(x;\lambda_0)$. Further, for each solution $u(x;\lambda)$ the sequence of functions

$$u_j(x;\lambda) = \frac{\partial^{j-1}u(x;\lambda)}{\partial\lambda^{j-1}} \qquad \text{for } j = 1, 2, \ldots, m \tag{2.2}$$

are independent and when evaluated at $\lambda = \lambda_0$ satisfy $L^m[y(x)] = 0$.

Example 1

Consider the constant coefficient linear ODE

$$L[y(x)] = \left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 1\right)y = y'' + y = \sin x \tag{2.3}$$

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Since $L[\sin x] = 0$ this linear ODE is being forced at resonance. Choosing $g(\lambda) = -\lambda^2$ and $\lambda_0 = 1$ we define $L_\lambda = \widehat{L} - g(\lambda)$; specifically

$$L_\lambda[y(x)] = \left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - g(\lambda)\right) y = \left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \lambda^2\right) y = y'' + \lambda^2 y \tag{2.4}$$

so that $L_{\lambda_0}[y(x)] = L_1[y(x)] = L[y(x)]$. We note that $\widehat{L}[y] = y''$, but do not use it. The homogeneous problem for the L_λ operator, $L_\lambda[y(x;\lambda)] = 0$, has a solution $u(x;\lambda) = \sin(\lambda x)$ which, when evaluated at $\lambda = \lambda_0 = 1$ is the forcing function in (2.3). Using this $u(x;\lambda)$ in (2.1) gives

$$\widehat{u}(x;\lambda) = \frac{1}{g'(\lambda)} \frac{\partial u(x;\lambda)}{\partial \lambda} = \left(\frac{1}{-2\lambda}\right) x \cos(\lambda x) \tag{2.5}$$

Evaluating this at $\lambda = \lambda_0 = 1$ yields the particular solution to (2.3): $\widehat{u}(x;\lambda_0) = -\dfrac{x\cos x}{2}$. Combining this with the solutions to the homogenous equation $L[y(x)] = 0$, yields the general solution to (2.3), that is $y(x) = c_1 \sin x + c_2 \cos x - \dfrac{x \cos x}{2}$.

Example 2

Consider the linear ODE

$$L^2[y(x)] = 0 \qquad \text{with} \qquad L[y(x)] = \left(x\frac{\mathrm{d}}{\mathrm{d}x} - b\right) y = xy' - by \tag{2.6}$$

Choosing $g(\lambda) = \lambda$ and $\lambda_0 = b$ we define

$$L_\lambda[y(x)] = \left(x\frac{\mathrm{d}}{\mathrm{d}x} - g(\lambda)\right) y = \left(x\frac{\mathrm{d}}{\mathrm{d}x} - \lambda\right) y = xy' - \lambda y \tag{2.7}$$

so that $L_{\lambda_0}[y(x)] = L_b[y(x)] = L[y(x)]$. We note that $\widehat{L}[y] = xy'$, but do not use it. The homogeneous problem for the L_λ operator, $L_\lambda[y(x;\lambda)] = 0$, has a solution $u(x;\lambda) = x^\lambda$. Using this function in (2.1) gives

$$\widehat{u}(x;\lambda) = \frac{1}{g'(\lambda)} \frac{\partial u(x;\lambda)}{\partial \lambda} = x^\lambda \ln x. \tag{2.8}$$

Evaluating the solutions $u(x;\lambda)$ and $\widehat{u}(x;\lambda)$ at $\lambda = \lambda_0 = b$ and taking a linear combination of them gives the general solution to (2.6):

$$y(x) = c_1 x^b + c_2 x^b \ln x \tag{2.9}$$

Example 3

Consider the constant coefficient ODE

$$L^3[y(x)] = 0 \qquad \text{with} \qquad L[y(x)] = y'' + \frac{1}{x} y' + y \tag{2.10}$$

Choosing $g(\lambda) = -\lambda^2$ and $\lambda_0 = 1$ we define

$$L_\lambda[y(x)] = y'' + \frac{1}{x} y' + \lambda^2 y \tag{2.11}$$

so that $L_{\lambda_0}[y(x)] = L_1[y(x)] = L[y(x)]$. The homogeneous problem for the L_λ operator, $L_\lambda[y(x;\lambda)] = 0$, has two linearly independent solutions $u_{11}(x;\lambda) = \mathrm{J}_0(\lambda x)$ and $u_{21}(x;\lambda) =$

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$\mathrm{Y}_0(\lambda x)$ where $\{J_n(x), Y_n(x)\}$ are Bessel functions. Using (2.2) yields the iteration of each of these functions

$$\begin{aligned} u_{12}(x;\lambda) &= \frac{\partial u_{11}(x;\lambda)}{\partial\lambda} = x\,\mathrm{J}_0'(\lambda x) = -x\,\mathrm{J}_1(\lambda x), \\ u_{22}(x;\lambda) &= \frac{\partial u_{21}(x;\lambda)}{\partial\lambda} = x\,\mathrm{Y}_0'(\lambda x) = -x\,\mathrm{Y}_1(\lambda x). \end{aligned} \tag{2.12}$$

Using (2.2) again gives the next iteration of these functions

$$\begin{aligned} u_{13}(x;\lambda) &= \frac{\partial u_{12}(x;\lambda)}{\partial\lambda} = -x^2\,\mathrm{J}_1'(\lambda x) = -\frac{x}{\lambda}\,\mathrm{J}_1(\lambda x) + x^2\,\mathrm{J}_2(\lambda x) \\ u_{23}(x;\lambda) &= \frac{\partial u_{22}(x;\lambda)}{\partial\lambda} = -x^2\,\mathrm{Y}_1'(\lambda x) = -\frac{x}{\lambda}\,\mathrm{Y}_1(\lambda x) + x^2\,\mathrm{Y}_2(\lambda x) \end{aligned} \tag{2.13}$$

Evaluating all the solutions found at $\lambda = \lambda_0 = 1$, taking a linear combination of them, and combining like terms gives the general solution to (2.10):

$$y(x) = c_1\,\mathrm{J}_0(x) + c_2\,\mathrm{Y}_0(x) + c_3 x\,\mathrm{J}_1(x) + c_4 x\,\mathrm{Y}_1(x) + c_5 x^2\,\mathrm{J}_2(x) + c_6 x^2\,\mathrm{Y}_2(x) \tag{2.14}$$

Notes

1. The information and examples in this section are from Willms [1] which, in turn, is based on Gouveia and Stone [2].
2. While the problems addressed by this method can be solved using variation of parameters, the current method is easier and faster.
3. For first order linear operators, explicit solutions to be written using this method (Willms [1, §3.8]). Consider $L = p(x)\frac{\mathrm{d}}{\mathrm{d}x} + q(x)$ with $p(x) \neq 0$ on an interval of interest. Let $y = u(x)$ be a solution to $L[y(x)] = 0$, that is $u(x) = \exp\left(\int_{x_0}^{x} \frac{-q(z)}{p(z)}\mathrm{d}z\right)$. Then, using $L_\lambda = L - \lambda$ and $\lambda_0 = 0$, the m independent solutions of $L^m[y(x)] = 0$ are $u_j(x) = u(x)\left(\int_{x_0}^{x} \frac{\mathrm{d}z}{p(z)}\right)^{j-1}$ for $j = 1, 2, \dots, m$.
   Applying this approach to $\left((x-a)(x-b)\dfrac{\mathrm{d}}{\mathrm{d}x} + x\right)^m y = 0$, the explicit solutions are
   $$u_j(x) = |x-a|^{\frac{a}{b-a}}|x-b|^{\frac{b}{b-a}}\left(\frac{1}{a-b}\ln\left|\frac{x-a}{x-b}\right|\right)^{j-1} \qquad \text{for } 1 \le j \le m \tag{2.15}$$
4. Willms [1] applies this method to
   $$\left(3\frac{\mathrm{d}^2}{\mathrm{d}x^2} - 6\frac{\mathrm{d}}{\mathrm{d}x} + 3\right)y = 3\left(\frac{\mathrm{d}}{\mathrm{d}x} - 1\right)^2 y = e^x \qquad \text{or} \qquad L_2[y] = 3L_1^2[y] = e^x. \tag{2.16}$$
   where $L_1[e^x] = 0$. He shows that the method fails if L_λ is defined relative to L_2 but is successful if L_λ is defined relative to L_1.

References

[1] A. R. Willms, "Resonantly Forced ODEs and Repeated Roots," *SIAM Review*, Vol. 66, Iss. 1, Feb 2024, Pages 149–160. `https://doi.org/10.1137/23M1545148` (Not cited.)

[2] B. Gouveia and H. A. Stone, "Generating Resonant and Repeated Root Solutions to Ordinary Differential Equations Using Perturbation Methods," *SIAM Review*, Vol. 64, Iss. 2, June 2022, Pages 485–499. `https://doi.org/10.1137/21M1395922` (Not cited.)

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3. Multiplicative DEs

Applicable to Second order multiplicative DEs of the following form. (This uses the multiplicative calculus and multiplicative derivative (y^*); on page 7 see notes #1 and #2.)

$$y^{**}\,(y^*)^{p(x)}\,(y)^{q(x)} = 1 \tag{3.1}$$

for the unknown positive function $y(x)$, where $p(x)$ and $q(x)$ are continuous.

Yields

In some cases, a solution.

Idea

Some of the ideas for ODEs (e.g., reduction of order) can be applied to multiplicative differential equations.

Procedure 1

Convert to a normal form: Start with equation (3.1). Let $v(x)$ satisfy $\dfrac{2v'}{v} + p = 0$ so that $v = \exp\left(-\dfrac{1}{2}\displaystyle\int p\,\mathrm{d}x\right)$. Then change variables

$$y(x) = u(x)^{v(x)}. \tag{3.2}$$

Using $y^* = (u^*)^v\;u^{(v')}$ and $y^{**} = (u^{**})^v\;(u^*)^{(2v')}\;(u)^{(v'')}$ results in a normal form

$$u^{**}\,u^{r(x)} = 1 \quad \text{where} \quad r(x) = -\frac{p'}{2} - \frac{p^2}{4} + q. \tag{3.3}$$

This normal form has removed the multiplicative first derivative term.

Example 1

The second order multiplicative DE

$$y^{**}\;(y^*)^{2x^2}\;(y)^{x^4+2x-4} = 1 \tag{3.4}$$

has $p(x) = 2x^2$ and $q(x) = x^4 + 2x - 4$; therefore $r(x) = -4$ in (3.3). The transformation in (3.2) with $v = \exp\left(-\frac{1}{2}\int p\,\mathrm{d}x\right) = \exp\left(-x^3/3\right)$ changes equation (3.4) to be in the normal form: $u^{**}\,u^{-4} = 1$ which has the solution $u = (c_1)^{\left(e^{-2x}\right)}\,(c_2)^{\left(e^{2x}\right)}$.

Procedure 2

Reduction of order: Start with equation (3.1). Suppose we know one solution, $y_1(x)$. We look for a second solution of the form $y_2(x) = (y_1(x))^{(\int u\,\mathrm{d}x)}$. This will be a solution when $u(x)$ satisfies

$$\frac{u'}{u} = -\left(\frac{2\ln y_1^*}{\ln y_1} + p\right) \qquad \text{or} \qquad u(x) = \frac{c}{(\ln y_1)^2}\exp\left(-\int_{x_0}^{x} p(z)\,\mathrm{d}z\right) \tag{3.5}$$

where c is a constant of integration. Combining the two solutions, the most general solution is then

$$y(x) = (y_1(x))^{c_1}\,(y_2(x))^{c_2} = (y_1(x))^{\left(c_1 + c_2\displaystyle\int \frac{e^{-\int_{x_0}^{x} p(z)\,\mathrm{d}z}}{(\ln y_1(x))^2}\,\mathrm{d}x\right)} \tag{3.6}$$

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Example 2

The second order multiplicative DE

$$y^{**}y^{\left(-2\sec^2 x\right)} = 1 \tag{3.7}$$

has a solution $y_1(x) = e^{\tan x}$ for $x > 0$. Using this solution with $p(x) = 0$ in (3.6) results in $y_2(x) = e^{-1-x\tan x}$. Hence, the general solution to (3.7) becomes $y(x) = e^{(c_1 \tan x - c_2 - c_2 x \tan x)}$.

Procedure 3

Convert to Riccati equation: Start with equation (3.1). Look for a solution of the form $y = \exp\left(\left(\int z^{\mathrm{d}x}\right)^{-1}\right)$. Using $y^* = z^{-\ln y}$ and $y^{**} = \left(\dfrac{z^{\ln z}}{z^*}\right)^{\ln y}$ in (3.1) results in $z^* z^{p(x)} e^{-q(x)} = z^{\ln z}$. Replacing z^* with $\exp(z'/z)$, taking logarithms, and setting $w = \ln z$ results in a Riccati equation for $w(x)$: $w' + p(x)w - q(x) = w^2$. Solving for $w(x)$, using $z = e^w$, and performing a multiplicative integration results in a solution for $y(x)$.

Example 3

The multiplicative DE $y^{**}\left(y^*\right)^{\frac{1}{x}}(y)^{-\frac{1}{x^2}} = 1$ can, using the above procedure, be converted to the Riccati equation $w' + \dfrac{1}{x}w + \dfrac{1}{x^2} = w^2$. This equation has the solution $w = \dfrac{1-2cx^2}{x+2cx^3}$ resulting in the solution $y(x) = \exp\left(\frac{1+2cx^2}{x}\right)$.

Notes

1. The multiplicative calculus was defined by Grossman and Katz [3]. Bashirov *et al.* [1] has a good summary with meaningful use cases. The general n^{th} other multiplicative DE is written as

$$\left(y^{*(n)}\right)^{a_n(x)}\left(y^{*(n-1)}\right)^{a_{n-1}(x)}\cdots\left(y^{**}\right)^{a_2(x)}\left(y^*\right)^{a_1(x)}(y)^{a_0(x)} = f(x) \tag{3.8}$$

   for the unknown positive function $y(x)$.

2. Multiplicative derivatives: Assume $\{f(x), g(x), h(x)\}$ are positive functions on an interval of interest and p is a positive constant. The multiplicate derivative of f is denoted f^*. Properties include:
   (a) $f^*(x) = \dfrac{\mathrm{d}^* f(x)}{\mathrm{d}x} = \lim_{h\to 0}\left[\dfrac{f(x+h)}{f(x)}\right]^{1/h} = \lim_{h\to 0}\left[1 + \left(\dfrac{f(x+h)-f(x)}{f(x)}\right)\right]^{1/h}$
   (b) $f^* = \exp\left(\dfrac{f'(x)}{f(x)}\right) = \exp\left((\ln f)'\right)$ and $f^{*(n)} = \exp\left((\ln f)^{(n)}\right)$
   (c) $(f^*)^* = f^{**} = f^{*(2)}$ and $f^{*(n)} = \left(f^{*(n-1)}\right)^*$
   (d) $(pf)^* = f^*$
   (e) $(fg)^* = f^* g^*$
   (f) $\left(\dfrac{f}{g}\right)^* = \dfrac{f^*}{g^*}$
   (g) $\left(f^h\right)^* = \left(f^*\right)^h f^{h'}$
   (h) $(f+g)^* = (f^*)^{\frac{f}{f+g}}(g^*)^{\frac{g}{f+g}}$
   (i) $(f \circ g)^* = [f^*(g)]^{g'}$

3. Multiplicative integrals: Assume $\{f(x), g(x)\}$ are positive bounded functions on an interval of interest and p is a positive constant. The multiplicate integral of f is denoted $\int_a^b f(x)^{\mathrm{d}x}$.
   (a) $\int_a^b f(x)^{\mathrm{d}x} = \exp\left(\int_a^b \ln f(x)\,\mathrm{d}x\right)$ so that $\int_a^b f(x)\,\mathrm{d}x = \ln \int_a^b \left(e^{f(x)}\right)^{\mathrm{d}x}$

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(b) $\displaystyle\int_a^b [f(x)g(x)]^{\mathrm{d}x} = \left(\int_a^b f(x)^{\mathrm{d}x}\right)\left(\int_a^b g(x)^{\mathrm{d}x}\right)$

(c) $\displaystyle\int_a^b f(x)^{\mathrm{d}x} = \left(\int_a^c f(x)^{\mathrm{d}x}\right)\left(\int_c^b f(x)^{\mathrm{d}x}\right)$

(d) $\displaystyle\int_a^b [f(x)^p]^{\mathrm{d}x} = \left(\int_a^b f(x)^{\mathrm{d}x}\right)^p$ (e) $\displaystyle\int_a^b \left(\frac{f(x)}{g(x)}\right)^{\mathrm{d}x} = \frac{\int_a^b f(x)^{\mathrm{d}x}}{\int_a^b g(x)^{\mathrm{d}x}}$

(f) $\displaystyle\int_a^b f^*(x)^{\mathrm{d}x} = \frac{f(b)}{f(a)}$

4. Values of multiplicative derivatives and integrals (for $a > 0$ and $b > 0$)

$f(x)$	$f^*(x)$	$\int f(x)^{\mathrm{d}x}$
a	1	a^x
x	$\exp\left(\frac{1}{x}\right)$	$\left(\frac{x}{e}\right)^x$
x^a	$\exp\left(\frac{a}{x}\right)$	$\left(\frac{x}{e}\right)^{ax}$
$ax+b$	$\exp\left(\frac{a}{ax+b}\right)$	$\frac{(b+ax)^{\frac{b}{a}+x}}{e^x}$
a^x	a	$a^{x^2/2}$
x^x	ex	$e^{-\frac{1}{4}x^2(1-2\ln x)}$

5. The first order differential equations $y^*(x) = f(x, y(x))$ and $y'(x) = y(x)\ln f(x, y(x))$ are equivalent.
6. The solution to $y^*(x) = y(x)$ with $y(0) = y_0$ is $y(x) = (y_0)^{e^x}$.
7. The solution to $y^*(x) = y(x)^{-c}$ with $y(0) = y_0$ is $y(x) = (y_0)^{e^{-cx}}$.
8. The general solution to $y^{**}(x) = \dfrac{1}{y(x)}$ is $y(x) = \exp(c_1 \sin x + c_2 \cos x)$.
9. Most of the content of this section is from Gülşen [2].
10. Yalçm and Dedeturk [4] describe how to solve multiplicative differential equations using transforms.

References

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APPROXIMATE METHODS

4. Magnus Expansion for Linear ODEs

Applicable to Linear matrix ODEs $\mathbf{y}'(t) = A(t)\mathbf{y}(t)$, specifically when the matrix does not commute at different values of the argument $A(t)A(s) \neq A(s)A(t)$.

Yields

A formal expansion of the solution which, in some cases, can be proven to converge.

Idea

Write the solution operator as an exponential of a matrix, where the matrix is a sum of terms. Each partial sum of the matrix creates an approximation of the evolution operator.

Procedure

Start with the general n-dimensional linear ODE

$$\mathbf{y}'(t) = A(t)\mathbf{y}(t) \tag{4.1}$$

as an initial value problem with $\mathbf{y}(0) = \mathbf{y}_0$ where $A(t)$ is a square matrix. We seek the time-evolution operator (also known as a propagator), a matrix $U(t)$ that satisfies $\mathbf{y}(t) = U(t)\mathbf{y}_0$. Of course $U(0) = I$, the identity matrix.

In some cases: (1) if $A(t)$ is a constant, (2) if $A(t)$ is a scalar, or (3) if $A(t)$ commutes with itself (i.e., $A(t)A(s) = A(s)A(t)$, for all t and s) then the solution to (4.1) is

$$U(t) = U_{\text{simple}}(t) = \exp\left(\int_0^t A(s)\,\mathrm{d}s\right) \tag{4.2}$$

More generally (i.e., applicable when none above three conditions are met) the time-evolution operator can be represented as $U(t) = e^{\Omega(t)}$ with $\Omega(t) = \sum_{k=1}^{\infty} \Omega_k(t)$. The Magnus expansion is a specific selection of the $\{\Omega_i\}$. The first few such terms are:

$$\Omega_1(t) = \int_0^t A(t_1)\,\mathrm{d}t_1, \qquad \Omega_2(t) = \frac{1}{2}\int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2\ [A(t_1), A(t_2)],$$

$$\Omega_3(t) = \frac{1}{6}\int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \int_0^{t_2} \mathrm{d}t_3\ \left(\Big[A_1, [A_2, A_3]\Big] + \Big[A_3, [A_2, A_1]\Big]\right),$$

$$\begin{aligned}\Omega_4(t) = \frac{1}{12}\int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \int_0^{t_2} \mathrm{d}t_3 \int_0^{t_3} \mathrm{d}t_4 \Bigg(&\Big[\left[[A_1, A_2], A_e\right], A_4\Big] \\ &+ \Big[A_1, \left[[A_2, A_3], A_4\right]\Big] + \Big[A_1, [A_2, [A_3, A_4]]\Big] + \Big[A_2, [A_3, [A_4, A_1]]\Big]\Bigg)\end{aligned} \tag{4.3}$$

Here $A_i = A(t_i)$ and $[B, C] = BC - CB$ is the commutator of the matrices B and C. A formulae is available for every term; Ω_k is a k-fold integral over $k-1$ nested commutators, see Magnus [5] and Arnal *et al.* [1].

Note that $U_{\text{simple}}(t)$ is the first approximation, $e^{\Omega_1(t)}$, in the Magnus expansion. If $A(t)$ commutes with itself, then all the higher order terms $\{\Omega_k \mid k = 2, 3, \dots\}$ vanish.

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Example

Consider the linear matrix ODE:

$$\mathbf{y}' = A(t)\mathbf{y} = \begin{bmatrix} 0 & t \\ 1 & 0 \end{bmatrix} \mathbf{y}. \tag{4.4}$$

To show that $A(t)$ is non-commutative we compute:

$$\begin{aligned} [A(t_1), A(t_2)] = A(t_1)A(t_2) - A(t_2)A(t_1) &= \begin{bmatrix} 0 & t_1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & t_2 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & t_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & t_1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} t_1 & 0 \\ 0 & t_2 \end{bmatrix} - \begin{bmatrix} t_2 & 0 \\ 0 & t_1 \end{bmatrix} = \begin{bmatrix} t_1 - t_2 & 0 \\ 0 & t_2 - t_1 \end{bmatrix} \neq 0. \end{aligned} \tag{4.5}$$

For this problem, the time evolution operator can be determined exactly. The solution to (4.4) can be written in terms of the Airy functions $\{\text{Ai}, \text{Bi}\}$ and their derivatives. The solution that satisfies $U(0) = I$ is:

$$\begin{aligned} U(t) &= \begin{bmatrix} m_1 \left(\text{Ai}'(t) - a_1 \text{Bi}'(t) \right) & m_2 \left(\text{Bi}'(t) - a_2 \text{Ai}'(t) \right) \\ m_1 \left(\text{Ai}(t) - a_1 \text{Bi}(t) \right) & m_2 \left(\text{Bi}(t) - a_2 \text{Ai}(t) \right) \end{bmatrix} \\ &= \begin{bmatrix} 1 + \frac{t^3}{3} + \frac{t^6}{72} + \dots & \frac{t^2}{2} + \frac{t^5}{30} + \frac{t^6}{1440} + \dots \\ t + \frac{t^4}{12} + \frac{t^2}{504} + \dots & 1 + \frac{t^3}{6} + \frac{t^6}{180} + \dots \end{bmatrix} \end{aligned} \tag{4.6}$$

for $\{m_1 = 3^{-2/3}\Gamma[-2/3],\ m_2 = 3^{1/6}\Gamma[2/3]/2,\ a_1 = 3^{-1/2},\ a_2 = -3^{1/2}\}$. The Taylor series around $t = 0$ is shown for later comparison.

We can compare this exact solution to the Magnus expansion. First we compute the first few $\{\Omega_i\}$:

$$\Omega_1(t) = \begin{bmatrix} 0 & \frac{t^2}{2} \\ t & 0 \end{bmatrix}, \quad \Omega_2(t) = \begin{bmatrix} \frac{t^3}{12} & 0 \\ 0 & -\frac{t^3}{12} \end{bmatrix}. \quad \Omega_3(t) = \begin{bmatrix} 0 & -\frac{t^5}{120} \\ 0 & 0 \end{bmatrix}, \quad \Omega_4(t) = \begin{bmatrix} -\frac{t^6}{360} & 0 \\ 0 & \frac{t^6}{360} \end{bmatrix}. \tag{4.7}$$

Using these, successive approximations of the $U(t)$ operator are obtained by taking partial sums of the $\{\Omega_i\}$ terms. For each of these, we find the Taylor series around $t = 0$ to compare against the exact result in (4.6).

$$\begin{aligned} U(t) \approx e^{\Omega_1} &= \exp\left(\begin{bmatrix} 0 & \frac{t^2}{2} \\ t & 0 \end{bmatrix} \right) = \begin{bmatrix} \cosh\lambda & \left(\frac{\lambda}{2}\right)^{1/3} \sinh\lambda \\ \left(\frac{2}{\lambda}\right)^{1/3} \sinh\lambda & \cosh\lambda \end{bmatrix} \qquad \text{with } \lambda = \sqrt{\frac{t^3}{2}} \\ &= \begin{bmatrix} 1 + \frac{\lambda^2}{2} + \dots & \frac{\lambda^{4/3}}{2^{1/3}} + \frac{\lambda^{10/3}}{6\,2^{1/3}} + \dots \\ 2^{1/2}\lambda^{2/3} + \frac{\lambda^{8/3}}{3\,2^{2/3}} + \dots & 1 + \frac{\lambda^2}{2} + \dots \end{bmatrix} \\ &= \begin{bmatrix} 1 + \frac{t^3}{4} + \frac{t^6}{96} + \dots & \frac{t^2}{2} + \frac{t^5}{24} + \frac{t^8}{960} + \dots \\ t + \frac{t^4}{12} + \frac{t^7}{488} + \dots & 1 + \frac{t^3}{4} + \frac{t^6}{96} + \dots \end{bmatrix} \\ U(t) \approx e^{\Omega_1 + \Omega_2} &= \begin{bmatrix} 1 + \frac{t^3}{4} + \frac{t^6}{48} + \dots & \frac{t^2}{2} + \frac{t^5}{24} + \frac{7t^8}{4320} + \dots \\ t + \frac{t^4}{12} + \frac{7t^7}{2160} + \dots & 1 + \frac{t^3}{6} + \frac{t^6}{144} + \dots \end{bmatrix} \\ U(t) \approx e^{\Omega_1 + \Omega_2 + \Omega_3} &= \begin{bmatrix} 1 + \frac{t^3}{3} + \frac{t^6}{60} + \dots & \frac{t^2}{2} + \frac{t^5}{30} + \frac{t^8}{4320} + \dots \\ t + \frac{t^4}{12} + \frac{t^7}{540} + \dots & 1 + \frac{t^3}{6} + \frac{t^6}{360} + \dots \end{bmatrix} \\ U(t) \approx e^{\Omega_1 + \Omega_2 + \Omega_3 + \Omega_4} &= \begin{bmatrix} 1 + \frac{t^3}{3} + \frac{t^6}{72} + \dots & \frac{t^2}{2} + \frac{t^5}{30} + \frac{t^8}{4320} + \dots \\ t + \frac{t^4}{12} + \frac{t^7}{540} + \dots & 1 + \frac{t^3}{6} + \frac{t^6}{180} + \dots \end{bmatrix} \end{aligned} \tag{4.8}$$

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These results show that, for this problem, using three Ω terms yields the correct first two terms in the Taylor expansion. At least five terms are needed to obtain the next term in the Taylor expansion.

Notes

1. A good review of the Magnus expansion (up until 2008) is in Blanes *et al.* [2].
2. There is also a Magnus expansion for stochastic ODEs, see Kamm *et al.* [3]
3. A numerical application of the Magnus expansion to nonlinear ODEs is in Krull and Minion [4].
4. The Magnus series will converge for $t \in [0, T)$ if $\int_0^T ||A(s)||_2 \, \mathrm{d}s < \pi$, where $||\cdot||_2$ is the matrix norm; see Moan and Niesen [6].
5. The Magnus series expansion maintains useful attributes of the exact solution. For example, in classical mechanics the symplectic nature of the solution is preserved at every order of the Magnus expansion.
6. There are several ways to compute a matrix exponential (see Molar and VanLoan [7]), one way is to use the definition of the exponential function. For example, in (4.8) we need to find $e^{\Omega_1(t)} = \sum_{n=0}^{\infty} \frac{\Omega_1(t)^n}{n!}$. This can be readily evaluated since $\Omega_1(t) = \begin{bmatrix} 0 & \frac{t^2}{2} \\ t & 0 \end{bmatrix}$ has a special structure. Consider the matrix $M = \left[\begin{smallmatrix} 0 & a \\ b & 0 \end{smallmatrix}\right]$. We find that $M^2 = abI$, $M^3 = abM$ and, in general,
$$M^n = \begin{cases} (ab)^{n/2} I & \text{if } n \text{ is even,} \\ (ab)^{(n-1)/2} M & \text{if } n \text{ is odd.} \end{cases} \tag{4.9}$$
This leads to
$$\begin{aligned} e^M &= \sum_{n=0}^{\infty} \frac{M^n}{n!} = \sum_{k=0}^{\infty} \frac{M^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{M^{2k+1}}{(2k+1)!} \\ &= \left(\sum_{k=0}^{\infty} \frac{(ab)^k}{(2k)!} \right) I + \left(\sum_{k=0}^{\infty} \frac{(ab)^k}{(2k+1)!} \right) M = \cosh\left(\sqrt{ab}\right) I + \frac{\sinh\left(\sqrt{ab}\right)}{\sqrt{ab}} M \end{aligned} \tag{4.10}$$
Defining $\lambda = \sqrt{ab} = \sqrt{t^3/2}$ results in the value of $e^{\Omega_1(t)}$ in (4.8).
7. The MATLAB command `expm` and the Mathematica command `MatrixExp` compute a matrix exponential.

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