# Polya theory
# (With a radar application motivation)

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http://www.az-tec.com/zwillinger/talks/20010405/

http://nesystemsengineering.rsc.ray.com/training/memo.polya.pdf

# Abstract

*This lecture will show how Polya theory can be used in counting objects, which is often the design basis for statistical tests. Specifically, Polya theory determines the number of distinct equivalence classes of objects. It can also give counts for specific types of patterns within equivalence classes. While sounding esoteric, this simple-to-apply technique easily counts non-isomorphic (i.e., different) graphs, and many other combinatorial structures. An application to radar filtering of hard-limited data will be used to motivate the topic.*

# Outline

- Introduction
- Who am I? Why am I talking?
- Talk is about combinatorial counting method
- Problem motivation
  - Switching functions
  - Radar filtering
  - Necklaces: using 3 beads of 3 different colors, how many different necklaces are there?
- Polya theory
  - Permutation groups
  - Burnside's Lemma

# Daniel Zwillinger

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- Education: MIT, Caltech
- Joined Raytheon in 2001 (Sr Principal Engineer)
- Formerly at: RPI, JPL, BBN, IDA, MITRE, Exxon, Sandia Labs, Ironbridge Networks, consulting
- Broad expertise in applied mathematics (continuous and discrete)
- Books on: differential equations, statistics, integration methods, table of integrals
- Editor-in-chief of 30$^{\text{th}}$ edition of CRC's *Standard Mathematical Tables and Formulae*
- Home page: `http://www.az-tec.com/zwillinger`

# Combinatorial problems

- Combinatorial optimization
  - How to define a problem?
  - How many solutions are there?
  - How to identify the solutions?
  - How to find the best solution?
- Today's focus is on the number of solutions
  - Standard counting problems (next 2 slides)
  - Generating functions
  - Standard problems (assignment problem, etc.)
  - **Polya theory**

# Sample selection

## Choosing a sample of size $r$ from $m$ objects

| Order counts? | Repetitions allowed? | The sample is called an | Number of ways to choose the sample |
|---|---|---|---|
| No | No | $r$-combination | $C(m, r)$ |
| Yes | No | $r$-permutation | $P(m, r)$ |
| No | Yes | $r$-combination with replacement | $C^R(m, r)$ |
| Yes | Yes | $r$-permutation with replacement | $P^R(m, r)$ |

$$C(m, r) = \binom{m}{r} = \frac{m!}{r!(m-r)!}$$

$$P(m, r) = (m)_r = \frac{m!}{(m-r)!}$$

$$C^R(m, r) = C(m+r-1, r) = \frac{(m+r-1)!}{r!(m-1)!}$$

$$P^R(m, r) = m^r$$

# Balls into cells

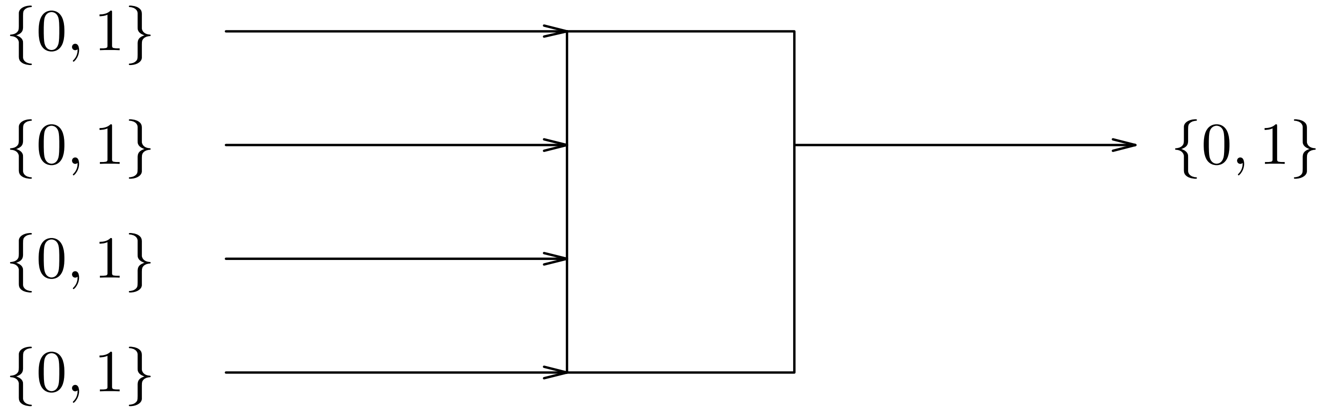
| Distinguish the cells | Distinguish the balls | Can cells be empty | Number of ways to place $n$ balls into $k$ cells |
|---|---|---|---|
| Yes | Yes | Yes | $k^n$ |
| Yes | Yes | No | $k!\left\{ n \atop k \right\}$ |
| Yes | No | Yes | $C(k+n-1, n) = \binom{k+n-1}{n}$ |
| Yes | No | No | $C(n-1, k-1) = \binom{n-1}{k-1}$ |
| No | Yes | Yes | $\left\{ n \atop 1 \right\} + \left\{ n \atop 2 \right\} + \cdots + \left\{ n \atop k \right\}$ |
| No | Yes | No | $\left\{ n \atop k \right\}$ |
| No | No | Yes | $p_1(n) + p_2(n) + \cdots + p_k(n)$ |
| No | No | No | $p_k(n)$ |

$\left\{ n \atop k \right\}$ is a Stirling cycle number

$p_k(n)$ is the number of partitions of $n$ into $k$ parts

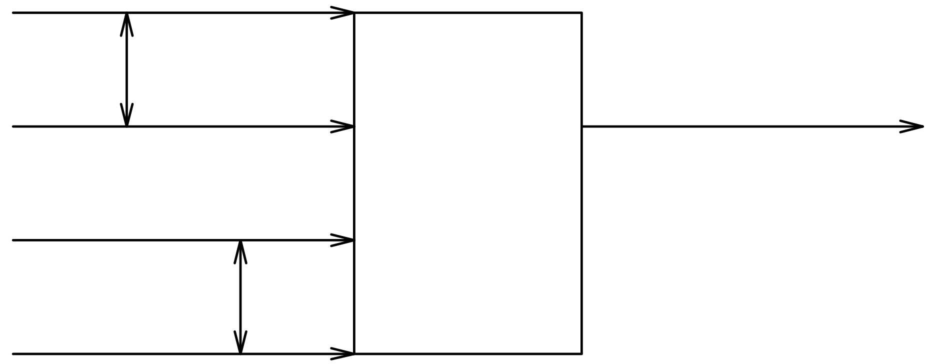
# Example: Switching functions (I)

- A switching function (Boolean function) assigns a 0 or 1 to each bit string of length $n$




- There are $2^{2^n}$ different switching functions (For $n = 4$ have 65,536 switching functions)

# Example: Switching functions (II)



- Allowing symmetries, there are only 3,984 *different* switching functions (see [2])!
- Allowing an inverter and symmetries, there are only 222 *different* switching functions (see [2])!
- Polya theory easily gives the number 222.

# Example: Radar filtering (I)

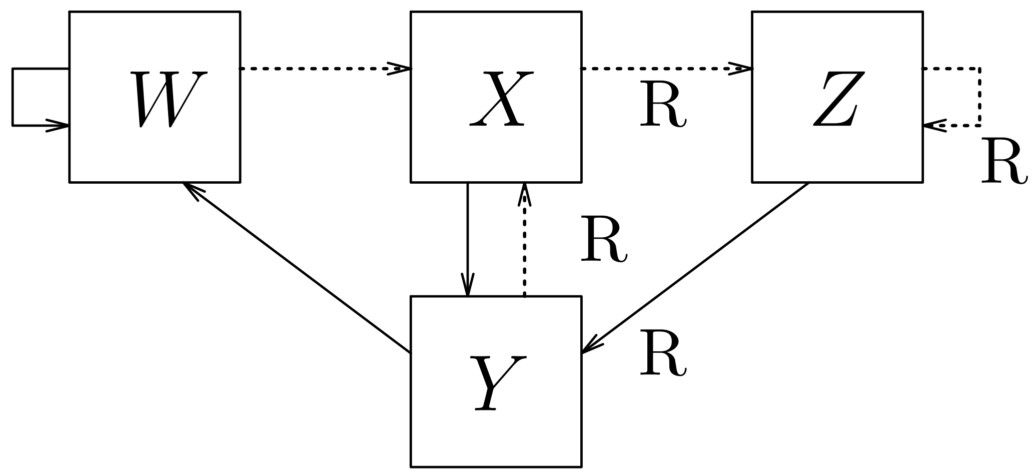
- D be a "detection", N is a "no detection"
- R is a "report", "–" is "no report"
- Sample filter result for "2 out of 3" filter

| | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| input | N | N | N | N | N | D | N | D | D | N | N | D | D | D | D | N | N | N |
| output | – | – | – | – | – | – | – | R | R | R | – | – | R | R | R | R | – | – |

- "2 out of 3" filter has 4 states:
  - State "W", last two input elements are "NN"
  - State "X", last two input elements are "ND"
  - State "Y", last two input elements are "DN"
  - State "Z", last two input elements are "DD"

# Example: Radar filtering (II)

Transition diagram is




"N" is shown as a solid line

"D" is shown as a dashed line

# Example: Radar filtering (III)

- For iid inputs (D has a probability of $p$) have transition matrix

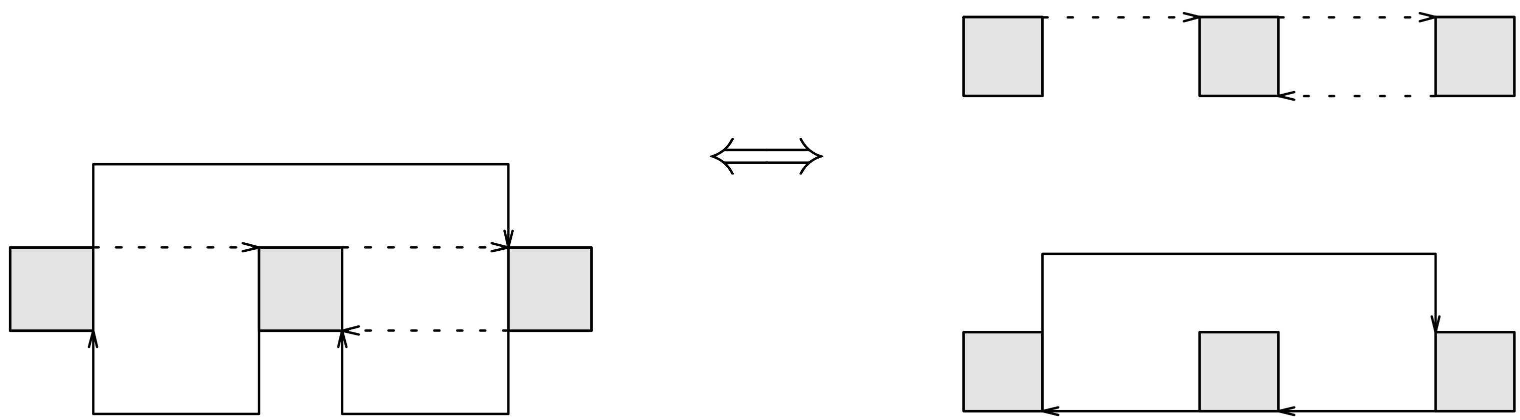
$$\text{From state}\quad \begin{array}{c} \\ W \\ X \\ Y \\ Z \end{array} \begin{array}{c} \begin{array}{cccc} W & X & Y & Z \end{array} \\ \begin{pmatrix} 1-p & p & 0 & 0 \\ 0 & 0 & 1-p & p \\ 1-p & p & 0 & 0 \\ 0 & 0 & 1-p & p \end{pmatrix} \end{array}$$

- Usual eigen-analysis yields steady state distribution:

| state | W | X | Y | Z |
|---|---|---|---|---|
| probability | $(1-p)^2$ | $p(1-p)$ | $p(1-p)$ | $p^2$ |

# Example: Radar filtering (IV)

- Hence, probability of an "R" is $P_{\mathrm{D}_{2\text{ of }3}} = p^2(3-2p)$
- For a given $P_{\mathrm{FA}}$ want to maximize $P_{\mathrm{D}_{2\text{ of }3}}$
- Can use input distributions other than iid ...
- Concept: enumerate all possible filters, pick the best
- How many are there? (Look at decomposition)



# Counting transition diagrams

- Need to know the number of directed graphs with out-degree one for different numbers of vertices.
- Find (using Polya theory)

| number of states $S$ | 2 | 3 | 4 |
|---|---|---|---|
| naive counting $S^{2S}$ | 16 | 729 | 65536 |
| number of actual filters | 10 | 129 | 2836 |

- Use Sloane [4] to identify sequence A054050: $\{10, 129, 2836, 83061, 3076386, 136647824, \ldots\}$: "Number of non-isomorphic binary n-state automata" [5]

# Permutation groups: Fast review

- Consider permutations of $\{1, 2, 3, \ldots, n\}$
- $\pi = \left(\begin{smallmatrix} 1 & 2 & \cdots & n \\ a_1 & a_2 & \cdots & a_n \end{smallmatrix}\right) = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}$
  means $\{1 \to a_1,\ 2 \to a_2,\ \ldots,\ n \to a_n\}$.
- Operation: $\pi_1 \circ \pi_2$ means: permute by $\pi_2$ then by $\pi_1$
- $\{\pi_1, \pi_2, \ldots, \pi_n\}$ can be closed to form a group $G$
- "$a$" is invariant if $\pi(a) = a$;
  "Inv($\pi$)" is number of elements invariant under $\pi$
- Write $\pi$ as unique product of disjoint cycles;
  "cyc($\pi$)" is number of cycles in $\pi$
- An equivalence relation $S$ (on $G$) is defined by
  "$a\ S\ b$" iff there exists a $\pi \in G$ such that $\pi(a) = b$

# Permutation groups: Example

If $G = \{\pi_1, \pi_2\}$ with

- $\pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 3 \end{pmatrix} \begin{pmatrix} 2 \end{pmatrix}$
- $\pi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 4 \end{pmatrix}$

Then

- $|G| = 2$
- $\pi_2 \circ \pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} \neq \pi_1 \circ \pi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$
- $\mathrm{Inv}(\pi_1) = 1 \qquad \mathrm{Inv}(\pi_2) = 0$
- $\mathrm{cyc}(\pi_1) = 2 \qquad \mathrm{cyc}(\pi_2) = 2$

# Burnside's Lemma

If

$A =$ set of elements

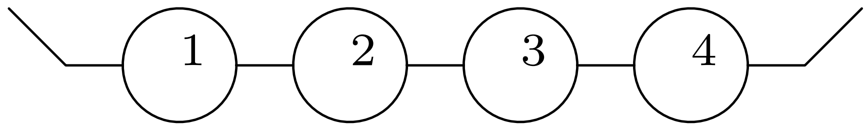
$G =$ permutation group on $A$

$S =$ equivalence relation on $A$ induced by $G$

Then the number of equivalence classes in $S$ is

$$\frac{1}{|G|} \sum_{\pi \in G} \mathrm{Inv}(\pi)$$

# Example: 4-bead necklaces




- $G = \{\pi_1, \pi_2\} = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \right\}$
- $|G| = 2$
- $\text{Inv}(\pi_1) = 4 \qquad \text{Inv}(\pi_2) = 0$
- Burnside's lemma: $\frac{1}{2}(4 + 0) = 2$ equivalence classes
- The 2 equivalence classes are $\{1, 4\}$ and $\{2, 3\}$

# Example: Necklaces with colored beads (I)

- Example: have 3 beads and 2 bead colors (B, W)
- $G = \{\pi_1, \pi_2\} = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \right\}$
- Colorings are:

$$\begin{array}{ll} a = \{W, W, W\} & e = \{B, W, W\} \\ b = \{W, W, B\} & f = \{B, W, B\} \\ c = \{W, B, W\} & g = \{B, B, W\} \\ d = \{W, B, B\} & h = \{B, B, B\} \end{array}$$

- Mappings: under $\pi_2$ have $b \Longleftrightarrow e$ and $d \Longleftrightarrow g$

## **Example:** Necklaces with colored beads (II)

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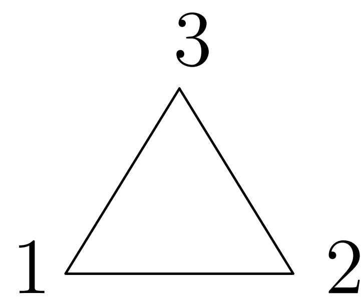
- $G^* = \{\pi_1^*, \pi_2^*\}$
- $\pi_1^* = \begin{pmatrix} a & b & c & d & e & f & g & h \\ a & b & c & d & e & f & g & h \end{pmatrix}$
- $\pi_2^* = \begin{pmatrix} a & b & c & d & e & f & g & h \\ a & e & c & g & b & f & d & h \end{pmatrix}$
- $|G| = |G^*| = 2$
- $\text{Inv}(\pi_1) = 3$, $\text{Inv}(\pi_2) = 1$: Burnside $\Longrightarrow \frac{1}{2}(3+1) = 2$
- $\text{Inv}(\pi_1^*) = 8$, $\text{Inv}(\pi_2^*) = 4$: Burnside $\Longrightarrow \frac{1}{2}(8+4) = 6$
- The 6 equivalence classes are:
  $\{a\}$, $\{b, e\}$, $\{c\}$, $\{d, g\}$, $\{f\}$, $\{h\}$
- Note: 4 classes of 1 element, 2 classes of 2 elements

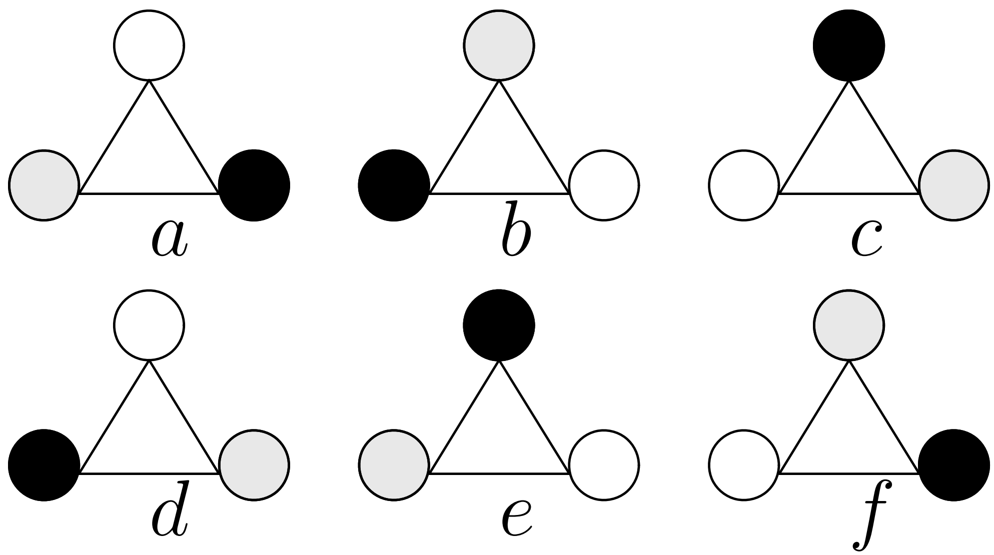
# Example: Rotations of an equilateral triangle




- $G = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \right\}$
- $|G| = 3$
- $\mathrm{Inv}(\pi_1) = 3$, $\mathrm{Inv}(\pi_2) = 0$, $\mathrm{Inv}(\pi_3) = 0$
- Burnside lemma: $\frac{1}{3}(3 + 0 + 0) = 1$ equivalence class
- The single equivalence class is $\{1, 2, 3\}$

## Example: Rotations of an equilateral triangle with colored beads (I)

- Three different colored beads (Black, White, Gray) on an equilateral triangle. Allow rotations.



# Example: Rotations of an equilateral triangle with colored beads (II)

- $G^* = \{\pi_1^*, \pi_2^*, \pi_3^*\} \qquad |G^*| = 3$
- $\pi_1^* = \begin{pmatrix} a & b & c & d & e & f \\ a & b & c & d & e & f \end{pmatrix} \qquad \text{Inv}(\pi_1^*) = 6$
- $\pi_2^* = \begin{pmatrix} a & b & c & d & e & f \\ b & c & a & e & f & d \end{pmatrix} \qquad \text{Inv}(\pi_2^*) = 0$
- $\pi_3^* = \begin{pmatrix} a & b & c & d & e & f \\ c & a & b & f & d & e \end{pmatrix} \qquad \text{Inv}(\pi_3^*) = 0$
- Burnside lemma: $\frac{1}{3}(6+0+0) = 2$ equiv classes
- The equivalence classes are $\{a, b, c\}$ and $\{d, e, f\}$
- Note: If flips are allowed, then one equivalence class

  [Need to add $\pi_4 = (1\ 3\ 2)$, $\pi_5 = (2\ 1\ 3)$, $\pi_6 = (3\ 2\ 1)$]

# Special case of Polya's Theorem

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If

$$G = \text{group of permutations of } D = \{\pi_1, \pi_2, \ldots\}$$

$$C = \text{set of } m \text{ colors}$$

$$S(D, C) = \text{set of colorings of } D \text{ using colors in } C$$

Then the number of distinct colorings in $S(D, C)$ is

$$\frac{1}{|G|} \sum_{\pi \in G} m^{\text{cyc}(\pi)}$$

# Example: Necklaces with colored beads (III)

- Assume $k$ beads
- $G = \{\pi_1, \pi_2\}$
- $\pi_1 = (1)(2)(3)\cdots(k)$
- $\pi_2 = (1\ k)(2\ k-1)(3\ k-2)\cdots$
- $\text{cyc}(\pi_1) = k \qquad \text{cyc}(\pi_2) = \begin{cases} \frac{k}{2} & \text{if } k \text{ is even} \\ \frac{k+1}{2} & \text{if } k \text{ is odd} \end{cases}$
- Number of equivalence classes of colorings is

$$\begin{cases} \frac{1}{2}\left[m^k + m^{k/2}\right] & \text{if } k \text{ is even} \\ \frac{1}{2}\left[m^k + m^{(k+1)/2}\right] & \text{if } k \text{ is odd} \end{cases}$$

# **Example:** Necklaces with colored beads (IV)

Number of equivalence classes for various $k$ and $m$

| | number of beads $k$ | | | | |
|---|---|---|---|---|---|
| | 2 | 3 | 4 | 5 | 6 |
| $m = 2$ colors | 3 | $\boxed{6}$ | 10 | 20 | 36 |
| $m = 3$ colors | 6 | $\underline{18}$ | 45 | 185 | 378 |
| $m = 4$ colors | 10 | 40 | 136 | 544 | 2080 |

# Example: Rotations of an equilateral triangle with colored beads (III)

- Assume $m$ colors
- $G = \{\pi_1, \pi_2, \pi_3\}$ $\quad |G| = 3$
- $\pi_1 = (1)(2)(3)$ $\quad \text{cyc}(\pi_1) = 3$
- $\pi_2 = (123)$ $\quad \text{cyc}(\pi_2) = 1$
- $\pi_3 = (132)$ $\quad \text{cyc}(\pi_3) = 1$
- Number of equivalence classes of colorings is

$$\frac{1}{3}\left[m^3 + 2m\right]$$

- For $m = 3$ have 11 colorings
- Before: 2 colorings with one B, one W, and one G

# Permutation groups: Cycle index

- Assume $\pi$ has $\begin{cases} b_1 \text{ cycles of length } 1 \\ b_2 \text{ cycles of length } 2 \\ \vdots \\ b_k \text{ cycles of length } k \end{cases}$
- Encode each $\pi$ by $x_1^{b_1}\ x_2^{b_2}\ \ldots\ x_k^{b_k}$
- Note: $\sum_i b_i(\pi) = \mathrm{cyc}(\pi)$
- Cycle index of $G$ is normalized sum of encodings

$$P_G(x_1, x_2, \ldots, x_k) = \frac{1}{|G|} \sum_{\pi \in G} x_1^{b_1(\pi)}\ x_2^{b_2(\pi)}\ \cdots\ x_k^{b_k(\pi)}$$

# Polya's Theorem

Let color $c$ have "weight" $w(c)$. The pattern inventory describes the colorings, of a certain kind, as functions of the weights.

$$G = \text{group of permutations on } D = \{\pi_1, \pi_2, \ldots\}$$

$$C = \text{set of colors; } |C| = m$$

$$S(D,C) = \text{set of colorings of } D \text{ using colors in } C$$

Then the pattern inventory of colorings in $S(D,C)$ is

$$P_G\left(\sum_{c\in C} w(c), \sum_{c\in C} w^2(c), \sum_{c\in C} w^3(c), \ldots\right)$$

Note: for $w(c) = 1$ have $P_G(m, m, \ldots, m) = \dfrac{1}{|G|}\sum_{\pi\in G} m^{\mathrm{cyc}(\pi)}$

## Example: Necklaces with colored beads (V a)

- Assume $k$ beads
- $G = \{\pi_1, \pi_2\}$
- $\pi_1 = (1)(2)(3)\cdots(k)$
- $\pi_2 = (1\ k)(2\ k-1)(3\ k-2)\cdots$
- $b_1(\pi_1) = k$ and $0 = b_2(\pi_1) = b_3(\pi_1) = \ldots$
- If $k$ is even, then $b_2(\pi_2) = \frac{k}{2}$ and
  $0 = b_1(\pi_2) = b_3(\pi_2) = \ldots$
  If $k$ is odd, then $b_1(\pi_2) = 1$, $b_2(\pi_2) = \frac{k-1}{2}$ and
  $0 = b_3(\pi_2) = b_4(\pi_2) = \ldots$

## **Example:** Necklaces with colored beads (V b)

- Cycle index is ($x_i = \sum_{c \in C} w^n(c)$)

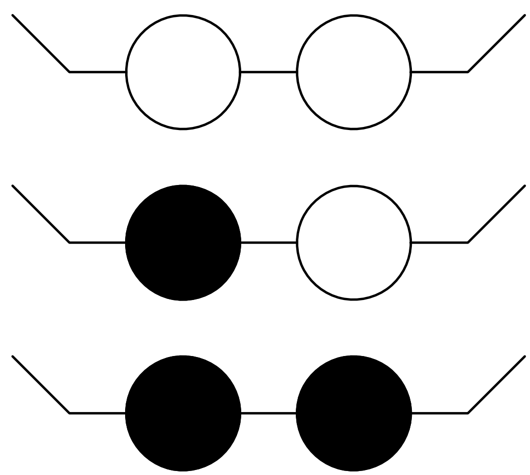
$$P_G(x_1, x_2) = \begin{cases} \frac{1}{2}\left[x_1^k + x_2^{k/2}\right] & \text{if } k \text{ is even} \\ \frac{1}{2}\left[x_1^k + x_1 x_2^{(k-1)/2}\right] & \text{if } k \text{ is odd} \end{cases}$$

- If $m = 1$ color (Black) available: $x_1 = B$, $x_2 = B^2$
  - For $k = 2$ have $P_G = \frac{1}{2}\left(B^2 + \left(B^2\right)^1\right) = B^2$
  - For $k = 3$ have $P_G = \frac{1}{2}\left(B^3 + B\left(B^2\right)^1\right) = B^3$
  - In general: $P_G = B^k$
  - Conclusion: there is a single coloring, and it has all black beads

## **Example:** Necklaces with colored beads (V c)

If $m = 2$ colors (Black, White) available:
$x_1 = (B + W)$, $x_2 = (B^2 + W^2)$

- For $k = 2$ have $P_G = \frac{1}{2}\left((B + W)^2 + \left(B^2 + W^2\right)^1\right)$
  $= W^2 + BW + B^2$

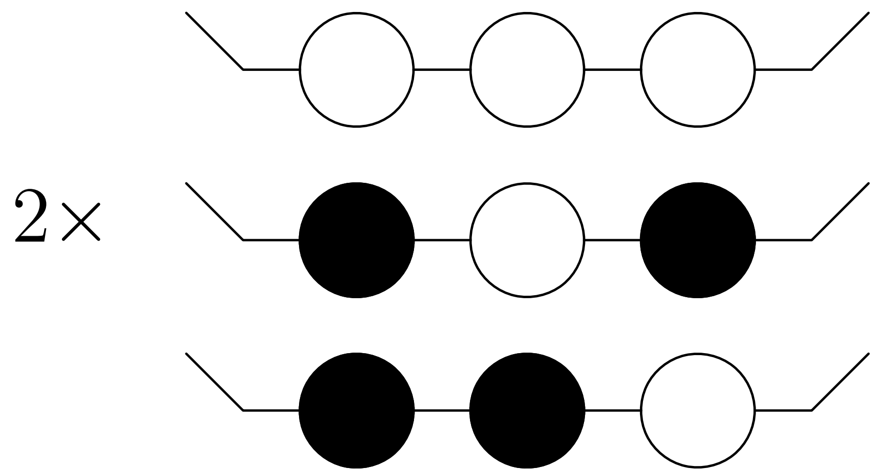


There are 3 distinct colorings

## **Example:** Necklaces with colored beads (V d)

If $m = 2$ colors (B, W) are available, then

- For $k = 3$: $P_G = \underbrace{W^3 + 2B^2W}_{\text{shown below}} + 2BW^2 + B^3$




There are 6 distinct colorings

- For even $k \geq 4$ have: $P_G = W^k + \frac{k}{2}BW^{k-1} + \ldots$

## **Example:** Necklaces with colored beads (V e)

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If $m = 3$ colors (Black, White, Shade) available:
$x_1 = (B + W + S)$, $x_2 = (B^2 + W^2 + S^2)$

- For $k = 2$ have 6 distinct colorings:

$$P_G = \frac{1}{2}\Big((B + W + S)^2 + (B^2 + W^2 + S^2)\Big)$$
$$= (B^2 + W^2 + S^2) + (BW + BS + SW)$$

- For $k = 3$ have 18 distinct colorings:

$$P_G = \frac{1}{2}\Big((B + W + S)^3 + (B + W + S)(B^2 + W^2 + S^2)\Big)$$
$$= (B^3 + W^3 + S^3) + 2(B^2W + W^2B + B^2S$$
$$+ S^2B + W^2S + S^2W) + 3BWS$$

# Example: Rotations of an equilateral triangle with colored beads (IV)

- $G = \{\pi_1, \pi_2, \pi_3\}, \quad |G| = 3$
- $\pi_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = (1)(2)(3) \qquad b_1(\pi_1) = 3$
- $\pi_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (2\ 3\ 1) \qquad b_3(\pi_2) = 1$
- $\pi_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = (3\ 1\ 2) \qquad b_3(\pi_3) = 1$
- Pattern inventory: $P_G = \frac{1}{3}\left(x_1^3 + 2x_3\right)$
- For $m = 3$ colors have 11 colorings:

$$P_G = \frac{1}{3}\left[(B + W + S)^3 + 2\left(B^3 + W^3 + S^3\right)\right]$$
$$= \left(B^3 + W^3 + S^3\right) + \cdots + \underline{\text{2BWS}} + \ldots$$

# Conclusion

- Many applications of Polya theory:
  - bracelets (rotations of an $n$-gon)
    For $n = 6$ find $P_G = \frac{1}{6}\left[x_1^6 + x_2^3 + 2x_3^2 + 2x_6\right]$
  - photomasks for chips
  - molecular structures
  - non-isomorphic graphs
  - seating arrangements
- Polya theory is more advanced than shown here
- Daniel Zwillinger can help in formulating and solving discrete and continuous mathematical problems (Sudbury 1-1-623, telephone 4–1660)

# References

## 1

[1] For pictures of colored necklaces, bracelets, unlabeled necklaces, etc, visit `http://www.theory.csc.uvic.ca/~cos/gen/neck.html` (be careful of terms!)

[2] Harvard Computation Laboratory Staff, *Synthesis of electronic computing and control circuits*, Harvard University Press, Cambridge, MA, 1951.

[3] Fred S. Roberts, *Applied Combinatorics*, Prentice-Hall, 1984.

[4] Sloane's on-line encyclopedia: `http://www.research.att.com/~njas/sequences/`

[5] F. Harary and E. Palmer, "Number of non-isomorphic binary n-state automata", *Graphical Enumeration*, 1973.

[6] Daniel Zwillinger, *CRC Standard Mathematical Tables and Formulae*, CRC, Boca Raton, FL, 1995.