Max-Plus Algebra

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The max-plus algebra $R_{\max}$ is an algebraic stucture using real numbers and two operations (called "addition" and "multiplication"); "addition" is the operation of taking a maximum, and "multiplication" is the "standard addition" operation. The max-plus algebra can be used in scheduling problems.

The max-plus semiring $R_{\max}$ is the set $R \cup \{-\infty\}$ with two operations called "plus" ($\oplus$) and "times" ($\otimes$) defined as follows:

- $a \oplus b = \max(a, b)$
- $a \otimes b = a + b$

For a and b in $R_{\max}$ the following laws apply:

1. Associativity
   - $(a \oplus b) \oplus c = a \oplus (b \oplus c)$
   - $(a \otimes b) \otimes c = a \otimes (b \otimes c)$
2. Commutativity
   - $a \oplus b = b \oplus a$
   - $a \otimes b = b \otimes a$
3. Distributivity
   - $(a \oplus b) \otimes c = a \otimes c \oplus b \otimes c$
4. Idempotency of $\oplus$
   - $a \oplus a = a$

Specially labeled elements are the "zero element" $\epsilon = -\infty$ and and the "unit element" $e = 0$.

- $\epsilon \oplus a = a$
- $\epsilon \otimes a = \epsilon$
- $e \otimes a = a$

Let A and B be matrices and let $[C]_{ij}$ be the ij^{th} element of matrix C. Then

- $[A \oplus B]_{ij} = [A]_{ij} \oplus [B]_{ij} = \max([A]_{ij}, [B]_{ij})$ when A and B have the same size
- $[A \otimes B]_{ij} = \bigoplus_{k=1}^{p}([A]_{ik} \otimes [B]_{kj}) = \max([A]_{i1} + [B]_{1j}, \dots, [A]_{ip} + [B]_{pj})$ when A has p columns and B has p rows.

Notes

1. $R_{\max}$ is not a group since not all elements have an additive inverse. (Only one element has an additive inverse, it is $-\infty$.)
2. The equation $a \oplus x = b$ need not have a unique solution.
   - If $a < b$, then $x = b$
   - If $a = b$, then x can be any value with $x \leq b$
   - If $a > b$, then there is no solution
3. In $R_{\max}$ matrix multiplication is associative.
4. Defining exponential as $a^n = \underbrace{a \otimes a \otimes \cdots \otimes a}_{n \text{ times}} = \underbrace{a + a + \cdots + a}_{n \text{ times}} = na$ we find $a^x \otimes a^y = a^{x+y}$ and $(a^x)^y = a^{xy}$, as in ordinary arithmetic.
5. Let A be a square matrix and let $(\lambda, \mathbf{x})$ satisfy $A\mathbf{x} = \lambda\mathbf{x}$ where λ is a real number and $\mathbf{x}$ is a conformable vector with at least one entry not equal to $-\infty$. Then λ is an eigenvalue of A with eigenvector $\mathbf{x}$.
   - If $(\lambda, \mathbf{x})$ is an eigenpair of A, then so is $(\lambda, c \otimes \mathbf{x})$ for any c.

6. The *Kleene star* of the matrix A is the matrix $A^* = I + A + A^2 + \cdots$.
7. The *completed max-plus semiring* $\overline{R}_{\max}$ is the same as $R_{\max}$ with the addition of the element $+\infty$ and the convention $(-\infty) + (+\infty) = (+\infty) + (-\infty) = -\infty$
8. The *digraph* $\Gamma(A)$ associated with with an $n \times n$ matrix A from $\overline{R}_{\max}$ has n vertices with an arc from vertex i to vertex j when $A_{ij} \neq 0$.
   - Let the $(k-1)$-step walk W on the digraph $\Gamma(A)$ be the sequence of arcs: $(i_1, i_2), (i_2, i_3), \ldots (i_{k-1}, i_k)$. Then the walk's *length* is $|W| = k - 1$ and the walk's *weight* is $|W|_A = A_{i_1 i_2} A_{i_2 i_3} \cdots A_{i_{k-1} 2 i_k}$
   - The matrix A is said to be *irreducible* if $\Gamma(A)$ is strongly connected.

Examples

Let $A = \begin{bmatrix} 10 & -\infty \\ 5 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 8 & 2 \\ 7 & 0 \end{bmatrix}$. Then

- Scalar multiplication of a matrix

$$5 \otimes A = \begin{bmatrix} 5 \otimes 10 & 5 \otimes (-\infty) \\ 5 \otimes 5 & 5 \otimes 3 \end{bmatrix} = \begin{bmatrix} 15 & -\infty \\ 10 & 8 \end{bmatrix}$$

- Matrix addition

$$A \oplus B = \begin{bmatrix} 10 \oplus 8 & -\infty \oplus 2 \\ 5 \oplus 7 & 3 \oplus 0 \end{bmatrix} = \begin{bmatrix} 10 & 2 \\ 7 & 3 \end{bmatrix}$$

- Matrix multiplication

$$A \otimes B = \begin{bmatrix} 10 \otimes 8 \oplus (-\infty) \otimes 7 & 10 \otimes 2 \oplus (-\infty) \otimes 0 \\ 5 \otimes 8 \oplus 3 \otimes 7 & 5 \otimes 2 \oplus 3 \otimes 0 \end{bmatrix} = \begin{bmatrix} 18 \oplus (-\infty) & 12 \oplus (-\infty) \\ 13 \oplus 10 & 7 \oplus 3 \end{bmatrix} = \begin{bmatrix} 18 & 12 \\ 13 & 7 \end{bmatrix}$$