General Relativity

Notation
Contributed by Dan Zwillinger.

- c speed of light
- G gravitational constant
- G_{ab} Einstein tensor
- g_{ab} metric tensor
- n dimension of the manifold
- R Ricci/curvature scalar
- R_{ab} Ricci curvature tensor

- $R^{\ell}{}_{ijk}$ Riemann tensor
- T_{ab} energy-momentum stress tensor
- Γ_{kij} Christoffel symbols of the first kind
- $\Gamma^{m}{}_{ij}$ Christoffel symbols of the second kind
- Λ cosmological constant
- ρ_{vac} vacuum constant

1. **Christoffel symbols**

$$\Gamma_{kij} = \frac{1}{2}\left(g_{ki,j} + g_{kj,i} - g_{ij,k}\right)$$

$$\Gamma^{m}{}_{ij} = g^{mk}\Gamma_{kij} = \frac{1}{2}g^{mk}\left(g_{ki,j} + g_{kj,i} - g_{ij,k}\right)$$

2. **Einstein's (field) equations**

$$G_{ab} = \frac{8\pi G}{c^4}T_{ab}$$

3. **Einstein tensor**

$$G_{ab} = R_{ab} + \left(\Lambda - \frac{1}{2}R\right)g_{ab}$$

4. **Ricci scalar** (also called curvature scalar)

$$R = R_{cd}g^{cd}$$

5. **Riemann tensor**

$$R^{\ell}{}_{ijk} = \frac{\partial}{\partial x^j}\Gamma^{\ell}{}_{ik} - \frac{\partial}{\partial x^k}\Gamma^{\ell}{}_{ij} + \sum_{s=i}^{n}\left(\Gamma^{\ell}{}_{js}\Gamma^{s}{}_{ik} - \Gamma^{\ell}{}_{ks}\Gamma^{s}{}_{ij}\right)$$

6. **Ricci tensor**

$$R_{ab} = R^{d}{}_{adb}$$

7. **Metrics**
   Note that $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$.

   (a) **Flat spacetime** in (t,x,y,z) coordinates

   $$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$

   (b) **Flat spacetime** in spherical (t,r,θ,ϕ) coordinates

   $$ds^2 = -c^2 dt^2 + dr^2 + r^2 d\Omega^2$$

   where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ is the standard metric on the 2-sphere.

   (c) **Schwarzschild metric** (M is a mass)

   $$ds^2 = -\left(1 - \frac{2GM}{r}\right)c^2 dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

8. **Vacuum constant**

$$\rho_{\mathrm{vac}} = \frac{\Lambda c^2}{8\pi G}$$

9. **Vacuum field equations**

$$R_{ab} = 0$$