General Relativity

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Notation

- c speed of light
- G gravitational constant
- G_{ab} Einstein tensor
- g_{ab} metric tensor
- n dimension of the manifold
- R Ricci/curvature scalar
- R_{ab} Ricci curvature tensor
- ${R^{\ell}}_{ijk}$Riemann tensor
- T_{ab} energy-momentum stress tensor
- Γ_{kij} Christoffel symbols of the first kind
- ${\Gamma^{m}}_{ij}$ Christoffel symbols of the second kind
- Λ cosmological constant
- ρ_{vac} vacuum constant

1. **Christoffel symbols**
$$\Gamma_{kij} = \frac{1}{2}\left(g_{ki,j} + g_{kj,i} - g_{ij,k}\right)$$
$${\Gamma^{m}}_{ij} = g^{mk}\Gamma_{kij} = \frac{1}{2}g^{mk}\left(g_{ki,j} + g_{kj,i} - g_{ij,k}\right)$$

2. **Einstein's (field) equations**
$$G_{ab} = \frac{8\pi G}{c^4}T_{ab}$$

3. **Einstein tensor**
$$G_{ab} = R_{ab} + \left(\Lambda - \frac{1}{2}R\right)g_{ab}$$

4. **Ricci scalar** (also called curvature scalar)
$$R = R_{cd}g^{cd}$$

5. **Riemann tensor**
$${R^{\ell}}_{ijk} = \frac{\partial}{\partial x^j}{\Gamma^{\ell}}_{ik} - \frac{\partial}{\partial x^k}{\Gamma^{\ell}}_{ij} + \sum_{s=i}^{n}\left({\Gamma^{\ell}}_{js}{\Gamma^{s}}_{ik} - {\Gamma^{\ell}}_{ks}{\Gamma^{s}}_{ij}\right)$$

6. **Ricci tensor**
$$R_{ab} = {R^{d}}_{adb}$$

7. **Metrics**
Note that $ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}$.
   (a) **Flat spacetime** in (t, x, y, z) coordinates
   $$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$
   (b) **Flat spacetime** in spherical (t, r, θ, ϕ) coordinates
   $$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2$$
   where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ is the standard metric on the 2-sphere.
   (c) **Schwarzschild metric** (M is a mass)
   $$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

8. **Vacuum constant**
$$\rho_{\text{vac}} = \frac{\Lambda c^2}{8\pi G}$$

9. **Vacuum field equations**
$$R_{ab} = 0$$