Coupled-analogues of Functions and Operations

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Many of the following coupled analogues are related to the usual q analogues via the substitution $\kappa = 1 - q$.

- Functions

  1. The coupled-logarithm is defined for $x > 0$ by: $\qquad \ln_\kappa(x) = \dfrac{x^\kappa - 1}{\kappa}$
     This function has the following properties:
     $$\begin{aligned} \lim_{\kappa \to 0} \ln_\kappa(x) &= \ln x \\ \ln_\kappa\left(x^a\right) &= a \ln_{a\kappa}(x) \\ \ln_\kappa\left(e_\kappa^x\right) &= x \end{aligned}$$
  2. The coupled-exponential is defined by:
     $$e_\kappa^x = \begin{cases} [1 + \kappa x]^{1/\kappa} & \text{when } 1 + \kappa x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$
     This function has the following properties:
     $$\begin{aligned} \lim_{\kappa \to 0} e_\kappa^x &= e^x \\ \left(e_\kappa\right)^a &= e_{\kappa/a}^{ax} \\ \exp_\kappa\left(\ln_\kappa(x)\right) &= x \end{aligned}$$
  3. The coupled-expectation of a function $f(x)$ given a probability distribution $p(x)$ is
     $$\mathrm{E}_\kappa\left[f(x)\right] = \frac{\int f(x) \, [p(x)]^{1-\kappa} \, dx}{\int [p(x)]^{1-\kappa} \, dx}$$
     This allows the following definitions: coupled-mean $= \overline{\mu}_\kappa = \mathrm{E}_\kappa\left[x\right]$, coupled-variance$= \overline{\sigma}_\kappa^2 = \mathrm{E}_\kappa\left[(x-\mu)^2\right]$. In the limit of $\kappa \to 0$ this reduces to the usual definition of mean and variance.

- Probability concepts

  1. The coupled-entropy (or Tsallis entropy) of the probabilities $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ is
     $$H_\kappa(\mathbf{p}) = \sum_{i=1}^N p_i \ln_\kappa\left(\frac{1}{p_i}\right) = -\sum_{i=1}^N p_i \ln_{-\kappa} p_i = -\sum_{i=1}^N p_i^{1-\kappa} \ln_\kappa p_i$$
  2. The coupled-entropy (or Tsallis entropy) of the continous probability distribution $f(x)$ is
     $$S_\kappa = \frac{1}{\kappa}\left(-1 + \int_{-\infty}^{\infty} f^{1-\kappa}(x) \, dx\right)$$
  3. The coupled-Gaussian probability density is given by
     $$G_\kappa\left(x; \overline{\mu}_\kappa, \overline{\sigma}_\kappa\right) = \frac{\sqrt{B_\kappa}}{C_\kappa} e_\kappa^{-B_\kappa (x - \overline{\mu}_\kappa)^2} = \frac{\sqrt{B_\kappa}}{C_\kappa}\left[1 + \kappa B_\kappa\left(x - \overline{\mu}_\kappa\right)^2\right]^{1/\kappa}$$
     where the width is $B_\kappa = [(2+\kappa)\overline{\sigma}_\kappa^2]^{-1}$ and the normalization factor C_κ is
     $$C_\kappa = \begin{cases} \sqrt{\dfrac{\pi}{\kappa}} \, \dfrac{\Gamma\left[\frac{1+\kappa}{\kappa}\right]}{\Gamma\left[\frac{2+3\kappa}{2\kappa}\right]} & \kappa > 0 \\ \sqrt{\pi} & \kappa = 0 \\ \sqrt{\dfrac{\pi}{-\kappa}} \, \dfrac{\Gamma\left[\frac{2+\kappa}{-2\kappa}\right]}{\Gamma\left[\frac{1}{-\kappa}\right]} & -2 < \kappa < 0 \end{cases}$$

This is the usual Gaussian when $\kappa = 0$, has compact support for $\kappa > 0$, decays asymptotically as a power law for $-2 < \kappa < 0$, and is equal to the Student-t distribution with $\nu = -\frac{2+\kappa}{\kappa}$ degrees of freedom.

4. The coupled-Gaussian random variable.

   A random variable X having a coupled-Gaussian distribution with coupled-mean $\overline{\mu}_\kappa$ and coupled-variance $\overline{\sigma}_\kappa^2$ is denoted $X \sim N_\kappa(\overline{\mu}_\kappa, \overline{\sigma}_\kappa)$.

   – A *standard coupled-Gaussian* is $N_\kappa(0,1)$

   – Coupled-Gaussian deviates: Given two independent random deviates $\{U_1, U_2\}$ from the uniform distribuion on $[0,1]$ two independent deviations from a standard coupled-Gaussian are given by

   $$Z_1 = \sqrt{-2\ln_{\kappa'}(U_1)}\sin(2\pi U_2)$$
   $$Z_2 = \sqrt{-2\ln_{\kappa'}(U_1)}\cos(2\pi U_2)$$

   where $\kappa' = \frac{2-\kappa}{2+\kappa}$. This is the same as the Box–Muller technique when $\kappa \to 0$.

- Operations

1. Coupled-addition is defined by: $x \oplus_\kappa y = x + y + \kappa xy$

   Note the properties:

   $$e_\kappa^x e_\kappa^y = e_\kappa^{x \oplus_\kappa y}$$
   $$\ln_\kappa(xy) = \ln_\kappa(x) \oplus \ln_\kappa(y)$$
   $$x \oplus y \oplus z = x + y + z + \kappa(xy + xz + yz) + \kappa^2 xyz$$

2. Coupled-subtraction is defined by: $x \ominus_\kappa y = \dfrac{x-y}{1+\kappa y}$

3. Coupled-multiplication is defined by: $x \otimes_\kappa y = (x^\kappa + y^\kappa - 1)^{1/\kappa}$

   Note the properties:

   $$e_\kappa^x \otimes_\kappa e_\kappa^y = e_\kappa^{x+y}$$
   $$\ln_\kappa(x \otimes_\kappa y) = \ln_\kappa(x) + \ln_\kappa(y)$$
   $$x_1 \otimes_\kappa x_1 \otimes_\kappa \cdots \otimes_\kappa x_n = \prod_{i=1}^{n}{}_\kappa\, x_i = (x_1^\kappa + x_2^\kappa + \cdots + x_n^\kappa - n + 1)^{1/\kappa}$$

4. Coupled-division is defined by: $x \oslash_\kappa y = (x^q - y^q + 1)^{1/\kappa}$

5. Differentiation rules:

   $$\frac{d}{dx} e_\kappa^{ax} = a \exp_{\frac{\kappa}{1-\kappa}}[(1-\kappa)ax] \qquad \text{when } \kappa \neq 1$$
   $$\frac{d^n}{dx^n} e_\kappa^{ax} = \left\{ a^n \prod_{i=1}^{n} [1-(i-1)\kappa] \right\} \exp_{\frac{\kappa}{1-n\kappa}}[(1-n\kappa)ax] \qquad \text{when } \kappa \neq 1, \frac{1}{2}, \ldots, \frac{1}{n}$$

6. Integration rules:

   $$\int e_\kappa^{ax}\, dx = \frac{1}{a(1+\kappa)} \exp_{\frac{\kappa}{1+\kappa}}[(1+\kappa)ax] + c_1 \qquad \text{when } \kappa \neq -1$$
   $$\underbrace{\int \cdots \int}_{n} e_\kappa^{ax}\, dx^n = \left[\frac{1}{a^n} \prod_{i=1}^{n} \frac{1}{1+i\kappa} \right] \exp_{\frac{\kappa}{1+n\kappa}}[(1+n\kappa)ax] + \sum_{i=1}^{n} c_i x^{i-1} \qquad \text{when } \kappa \neq -1, -\frac{1}{2}, \ldots, -\frac{1}{n}$$