Errata for the Third Edition of **Handbook of Differential Equations**

LAST UPDATED: July 30, 2018

NOTES:

1. The latest errata is available from http://www.mathtable.com/zwillinger/errata/.
2. The home page for this book is http://www.mathtable.com/hode/.
3. You can reach the author at ZwillingerBooks@gmail.com.

I thank everyone who has contacted me about mistakes in this book!

1. Section 1, **Definition of Terms**, page 3. The commutator example may be misunderstood. The correction is to change

   See Goldstein [6] for details.

   To

   Note that the "1", as an operator, represents the identity. Hence, the first term is (in a different notation) $(xd)(1+d) = xd + xd^2$; it is not xd^2. See Goldstein [6] for details.

   (Thanks to David Goldsmith for this correction.)

2. Section 6, **Classification of Partial Differential Equations**, page 35.

   (a) The equation between equations (6.3) and (6.4) currently has the line:

   $$u_{xy} = u_{\eta\eta}\eta_x\eta_y + \boxed{2}u_{\eta\zeta}(\eta_x\zeta_y + \eta_y\zeta_x) + u_{\zeta\zeta}\zeta_x\zeta_y + u_\eta\eta_{xy} + u_\zeta\zeta_{xy},$$

   which is incorrect, it should have been:

   $$u_{xy} = u_{\eta\eta}\eta_x\eta_y + u_{\eta\zeta}(\eta_x\zeta_y + \eta_y\zeta_x) + u_{\zeta\zeta}\zeta_x\zeta_y + u_\eta\eta_{xy} + u_\zeta\zeta_{xy},$$

   (b) The equation after equation (6.4) currently has the line:

   $$\overline{B} = A\zeta_x\eta_x + B(\zeta_x\eta_y + \zeta_y\eta_x) + 2C\zeta_y\eta_y,$$

   which is incorrect, it should have been (a "2" was missing)

   $$\overline{B} = 2A\zeta_x\eta_x + B(\zeta_x\eta_y + \zeta_y\eta_x) + 2C\zeta_y\eta_y,$$

   (Thanks to Hans Weertman for these corrections.)

3. Section 7, **Compatible Systems**, page 41, Special Case 3. The text for this special case is incorrect. It should be replaced with:

   > In the special case of $r = 1$, we have a system of m equations in m dependent variables. These equations do not require any side conditions.

   (Thanks to Rusty Humphrey for this correction.)

4. Section 11, **Fixed Point Existence Theorems**, page 54

   (a) The name "Schrauder" should be "Schauder"

   (b) The following reference should be added:

   > J. Schauder, "Der Fixpunktsatz in Funktionalräumen," **Studia Math.**, 2, (1930), 171–180.

   (Thanks to G. Friesecke for these corrections.)

5. Section 13, **Integrability of Systems**, page 65, Note number 11 contains "the sine–Gordan equation" when it should have "the sine–Gordon equation".

   (Thanks to Alain Moussiaux for this correction.)

6. Section 17, **Natural Boundary Conditions for a PDE**, page 77, The equation at the top of page 77, before equation (17.1) is now

$$J[\phi + h] - J[\phi] = \iint\limits_R \left\{ L_{\phi_t} h_t + L_{\phi_{x_j}} h_{x_j} + L_\phi \right\} dt\, d\mathbf{x} + O(||h||^2),$$

   This is incorrect, it should have been

$$J[\phi + h] - J[\phi] = \iint\limits_R \left\{ L_{\phi_t} h_t + L_{\phi_{x_j}} h_{x_j} + L_\phi \boxed{h} \right\} dt\, d\mathbf{x} + O(||h||^2),$$

   (Thanks to Zhuo Li for this correction.)

7. Section 27, **Canonical Forms**, page 118, reference number 2 is now

   > Bateman, H. *Partial Differential Equations of Mathematical Physics*, Dover Publications, New York, 1944.

   Which is incorrect. The reference should have been

   > Bateman, H. *Differential Equations*, Longmans, Green and Co., New York, 1926, pages 75–79.

   (Thanks to Ali Nejadmalayeri for this correction.)

8. Section 36, **Transformations of Second Order Linear ODEs – 1**, page 137, equation (36.6) is now $I(x) = \left(b - \frac{1}{4}a^2 - \frac{1}{2}\frac{da}{dx}\right)$, which is incorrect. It should be $I(x) = \left(b + \frac{1}{4}a^2 - \frac{1}{2}\frac{da}{dx}\right)$, (Thanks to Richard D. Rabbitt for this correction.)

9. Section 44.1.2, **Look-Up Technique**, page 169, the two equations

   (a) Painlevé–Ince – modified

   (b) Pinney

   are both missing the "$= 0$" that should at the end of each.

   (Thanks to Alain Moussiaux for these corrections.)

10. Section 44.1.3, **Look-Up Technique**, page 172, last equation before section 44.2, presently has

    $$y^{(m)} = a\boxed{xy^{-m/2}}$$

    This is incorrect, it should have been

    $$y^{(m)} = a\boxed{yx^{-m/2}}$$

    (Thanks to Flavio Noca for this correction.)

11. Section 50, **Clairaut's Equation**, page 216, the equation between (50.5) and (50.6) is now

    $$y''[2(xy' - \boxed{2})x - 2y'] = 0$$

    which is incorrect. This expression should be

    $$y''[2(xy' - y)x - 2y'] = 0$$

    (Thanks to Bruno Muratori for this correction.)

12. Section 53, **Contact Transformation**, page 227, the second equation in equation 53.7 has the form

    $$\cdots = \left(2X^3 - 3X\right)^{1/3}$$

    which is incorrect. This expression should be

    $$\cdots = \boxed{C}\left(2X^3 - 3X\right)^{1/3}$$

    (Thanks to Alain Moussiaux for this correction.)

13. Section 72, **Green's functions**, page 292, From above equation (72.9) to that equation the text is presently:

    Using the second method, we find the eigenvalues and eigenfunctions to be

    $$\lambda_n = \boxed{\frac{n\pi}{L}}, \qquad \phi_n(x) = \sin \lambda_n x = \sin\left(\frac{n\pi x}{L}\right),$$

    so that

    $$G(x;z) = \boxed{\frac{2L}{n\pi}} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi z}{L}\right).$$

    which is incorrect; the text should have been

    Using the second method, we find the eigenvalues and eigenfunctions to be

    $$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad \phi_n(x) = \sin \lambda_n x = \sin\left(\frac{n\pi x}{L}\right),$$

    so that

    $$G(x;z) = \sum_{n=1}^{\infty} \boxed{\left(-\frac{2L^2}{n^2\pi^2}\right)} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi z}{L}\right).$$

    (Thanks to Luis Alberto Fernandez for this observation.)

14. Section 79, **Integrating Functions**, page 326, note number 10, the following should be added:

    The general solution to $u_x = yu_y$ is $u = f(x + \log y)$, where f is an arbitrary function.

    (Thanks to Alain Moussiaux for this observation.)

15. Section 80, **Interchanging Dependent and Independent Variables**, page 327,

    (a) In Example 3, the nonlinear equation is given as "$y''(x-y)y'^3$", which is incorrect. It should have been "$y''(y-x)y'^3$".
    (Thanks to Alain Moussiaux for this correction.)

    (b) In Note number 2, the reference to Bender and Orszag should be section 1.5, not 1.6.
    (Thanks to James Dare for this correction.)

    (c) A better citation for reference number 3 is: McAllister, B. L. and Thorne, C.J. "Reverse differential equations and others that can be solved exactly", *Studies Appl. Math*, 6, 1952.
    (Thanks to Daniele Ritelli for this correction.)

16. Section 85, **Reduction of order**, page 354, note number 2 presently contains

> More generally, if $\{z_1(x), \dots, z_p(x)\}$ are linearly independent solutions of equation (85.6), then the substitution
>
> $$y(x) = \begin{bmatrix} z_1 & \dots & z_p & v \\ z_1' & \dots & z_p' & v' \\ \vdots & & \vdots & \vdots \\ z_1^{(p)} & \dots & z_p^{(p)} & v^{(p)} \end{bmatrix}$$
>
> reduces equation (85.7) to a linear ordinary differential equation of order $n-p$ for $v(x)$.

This should be changed to

> More generally, if $\{z_1(x), \dots, z_p(x)\}$ are linearly independent solutions of equation (85.6), then the substitution
>
> $$y(x) = \begin{bmatrix} z_1 & \dots & z_p & z \\ z_1' & \dots & z_p' & z' \\ \vdots & & \vdots & \vdots \\ z_1^{(p)} & \dots & z_p^{(p)} & z^{(p)} \end{bmatrix} \phi(x) \tag{1}$$
>
> where $\phi(x)$ need not be specified, reduces equation (85.6) to a linear ordinary differential equation of order $n-p$ for $y(x)$. The following explains why.
>
> With the above, $y(x)$ can be written in the form
>
> $$y(x) = A(x)z^{(p)} + B(x)z^{(p-1)} + \dots, \qquad A(x) \neq 0$$
>
> and its derivatives have the form
>
> $$y'(x) = A(x)z^{(p+1)} + \dots, \qquad y''(x) = A(x)z^{(p+2)} + \dots,$$
>
> These equations can be used to eliminate $\{z^{(p)}, \dots, z^{(n)}\}$ and (85.6) will take the form
>
> $$b_0 y^{(n-p)} + \dots + b_{n-p} y + V = 0 \tag{2}$$
>
> where V is linear in the $\{z, z', \dots, z^{(p-1)}\}$.
>
> We argue that $V \equiv 0$ as follows: Consider equation (2) as a differential equation of degree $p-1$ in z (via the V term). If $z = z_i$ (for any $i = 1, 2, \dots, p$) then $y = 0$ from equation (1). Hence, from equation (2) it must be that $V|_{z=z_i} = 0$. Hence $\{z_i\}_{i=1,2,\dots,p}$ is a collection of p linearly independent solutions to a differential equation of degree $p-1$; possible only if $V \equiv 0$.

(Thanks to Unal Goktas for this correction.)

17. Section 87, **Matrix Riccati Equations**, page 358. The second line in equation (87.4) is now

$$\frac{dy}{dt} = b(t)(y^2 - x^2) - 2a(t)xy \boxed{-} 2cy$$

Which is incorrect, it should have been

$$\frac{dy}{dt} = b(t)(y^2 - x^2) - 2a(t)xy + 2cy$$

(Thanks to both Peter Sherwood and Alain Moussiaux for this correction.)

18. Section 93, **Superposition**, page 373, the last line contains the equation

$$L[y] = y'' + a(x)y' + b(x) = f(x)$$

Which is incorrect. This should have been

$$L[y] = y'' + a(x)y' + b(x)\boxed{y} = f(x)$$

(Thanks to Young Kim for this correction.)

19. Section 96, **Vector Ordinary Differential Equations** pages 384-385, In note number 9 the second equation is incorrect. All the text after "Alternately, if the ..." should be deleted.

(Thanks to Frankie Liu for this correction.)

20. Section 106, **Inverse Scattering**, page 416, the **Applicable to** statement should have at the end

having the form of (106.2)

(Thanks to G. Friesecke for this observation.)

21. Section 106, **Inverse Scattering**, page 418, Note number 5 gives a Lax pair for the equation $u_t + u_{xx} - 2uu_x = 0$, which is not quite the Burger's equation. (Notice the minus sign before the last term.)

(Thanks to Bruno Muratori for this correction.)

22. Section 118, **Chaplygin's Method**, page 465, equations (118.5) and (118.6) and the surrounding text are now

    Then define $u_1(x)$ to be the solution of

    $$y' = M(x)y + N(x), \qquad y(x_0) = y_0. \tag{118.5}$$

    and define $v_1(x)$ to be the solution of

    $$y' = \widehat{M}(x)y + \widehat{N}(x), \qquad y(x_0) = y_0. \tag{118.6}$$

    Which is incorrect. This should have been (note that the definitions have been switched):

    Then define $v_1(x)$ to be the solution of

    $$y' = M(x)y + N(x), \qquad y(x_0) = y_0. \tag{118.5}$$

    and define $u_1(x)$ to be the solution of

    $$y' = \widehat{M}(x)y + \widehat{N}(x), \qquad y(x_0) = y_0. \tag{118.6}$$

    (Thanks to Bruno Van der Bossche for these corrections.)

23. Section 123, **Graphical Analysis: The Phase Plane**, pages 479, 480.

    (a) In the text for example 1 it says

    > ...The curve figure 123.2 is given by determinant= $(\text{trace})^2$; only centers can occur along this curve.

    which is incorrect; it should have said

    > ...The curve in figure 123.2 is given by determinant= $(\text{trace}/2)^2$. Centers occur along the curve defined by trace= 0.

    (b)

    (Thanks to Zhuo Li for these corrections.)

24. Section 136, **Monge's Method**, pages 523–524,

    (a) Equation (136.5) contains, in part

    $$\cdots = \boxed{\frac{\partial z}{\partial y}} + 6y$$

    which is incorrect. This expression should be

    $$\cdots = \boxed{\frac{\partial z}{\partial x}} + 6y$$

    (b) Equation (136.10) contains, in part

    $$\cdots + \psi\left(2\boxed{z} + y^2\right)$$

    which is incorrect. This expression should be

    $$\cdots + \psi\left(2\boxed{x} + y^2\right)$$

    (Thanks to Alain Moussiaux for this correction.)

25. Section 139, **Perturbation Method: Method of Averaging**, pages 532–533,

    (a) In equations (139.3) and (139.5) the last "cos" in each case should be a "sin".

    (b) The two equations in (139.9) are each missing a final closing parenthesis.

    (Thanks to Gerald Teschl for these corrections.)

26. Section 143, **Perturbation Method: Regular Perturbation**, page 554, equations (143.5 b) and (143.7 b) both have "$y_1(0) = 1$" which is incorrect; they should have been "$y_1(0) = 0$".

    (Thanks to Frank Scharf for this corrections.)

27. Section 145, **Picard Iteration**, page 561, note number one, the following should be added:

    > However, the successive approximations are guaranteed to converge to the true solution for all x sufficiently close to zero provided f is a continuously differentiable function.

    (Thanks to G. Friesecke for this observation.)

28. Section 148, **Soliton-Type Solutions**, pages 567–569,

    (a) In equation (148.3) the term cv_ζ should be $-cv_\zeta$.

    (b) In equation (148.4) the term $(v_\zeta)^2$ should be $\frac{1}{2}(v_\zeta)^2$.

    (c) An additional note should be added on page 569 to state

    > With the standard choice of $A = B = 0$, the solution to (148.4) can be solved in terms of elementary functions:
    >
    > $$v(x) = \frac{3c}{\sigma}\left(\operatorname{sech}\left(\frac{\sqrt{c}x}{2}\right)\right)^2$$

    (Thanks to G. Friesecke for these corrections.)

29. Section 180, **Runge–Kutta Methods**, pages 691, 696

(a) Equation (180.3) is missing some "h" terms. Presently there is:

$$\begin{aligned} k_1 &= f(x_0, y_0), \\ k_2 &= f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1), \\ k_3 &= f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2), \\ k_4 &= f(x_0 + h, y_0 + k_3). \end{aligned} \tag{3}$$

which is incorrect. It should have been:

$$\begin{aligned} k_1 &= f(x_0, y_0), \\ k_2 &= f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}\boxed{h}k_1), \\ k_3 &= f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}\boxed{h}k_2), \\ k_4 &= f(x_0 + h, y_0 + \boxed{h}k_3). \end{aligned} \tag{4}$$

(b) Note number 9 is incorrect and should be deleted.

30. Section 172, **Pseudospectral Method**, page 772, presently has:

$$\left.\frac{\partial u}{\partial x}\right|_{x=x_k} \simeq \frac{1}{3h}(u_{k+1} - u_{k-1}) - \frac{1}{6h}(u_{k+2} - u_{k-2}).$$

and

$$\left.\frac{\partial u}{\partial x}\right|_{x=x_k} \simeq \frac{1}{2h}(u_{k+1} - u_{k-1}) - \frac{1}{3h}(u_{k+2} - u_{k-2}) + \frac{1}{30h}(u_{k+3} - u_{k-3}).$$

and

$$\left.\frac{\partial u}{\partial x}\right|_{x=x_k} = \sum_{j=1}^{\infty} \frac{2(-1)^{j+1}}{jh}(u_{k+j} - u_{k-j}).$$

Which are all incorrect. They should have been:

$$\left.\frac{\partial u}{\partial x}\right|_{x=x_k} \simeq \frac{2}{3h}(u_{k+1} - u_{k-1}) - \frac{1}{12h}(u_{k+2} - u_{k-2}).$$

and

$$\left.\frac{\partial u}{\partial x}\right|_{x=x_k} \simeq \frac{3}{4h}(u_{k+1} - u_{k-1}) - \frac{3}{20h}(u_{k+2} - u_{k-2}) + \frac{1}{60h}(u_{k+3} - u_{k-3}).$$

and

$$\left.\frac{\partial u}{\partial x}\right|_{x=x_k} = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{jh}(u_{k+j} - u_{k-j}).$$

(Thanks to Didier Clamond for these corrections.)

NEW MATERIAL

If Dan Zwillinger were writing *Handbook of Differential Equations* today, additional sections that would be added include:

- Definition and Concepts

  1. **Classification of Equations**

     (a) Roman Kozlov, *The group classification of a scalar stochastic differential equation*, Journal of Physics A: Mathematical and Theoretical, Volume 43, Number 5, 12 January 2010, `http://iopscience.iop.org/article/10.1088/1751-8113/43/5/055202/meta`

     Lie's group classification of the stochastic differential equation $dx = f(t,x)\,dt + g(t,x)\,dW(t)$

Group dimension	Basis operators	Equation
0	No symmetries	$dx = f(t,x)dt + dW(t)$
1	$X_1 = \frac{\partial}{\partial t}$	$dx = f(x)dt + dW(t)$
2	$X_1 = \frac{\partial}{\partial t}, \quad X_2 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$	$dx = \frac{\alpha}{x}dt + dW(t), \quad \alpha \neq 0$
3	$X_1 = \frac{\partial}{\partial t}, \quad X_2 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial x}$	$dx = dW(t)$

- Exact Methods

  1. **Differential Constraints**

     (a) Oleg V Kaptsov and Alexey V Schmidt, *Linear determining equations, differential constraints and invariant solutions*, 31 Aug 2003, `https://arxiv.org/pdf/math-ph/0309001.pdf`

     (b) Boris Kruglikov, *Symmetry approaches for reductions of PDEs, differential constraints and Lagrange-Charpit method'*, `http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.520.5511&rep=rep1&type=pdf`

     (c) Peter J. Olver, *Direct Reduction and Differential Constraints*, `http://www-users.math.umn.edu/~olver/s_/dc.pdf`

  2. **Eigenvalue problems**

     (a) Donald S. Cohen, *An Integral Transform Associated with Boundary Conditions Containing an Eigenvalue Parameter*, SIAM J. Appl. Math., Volume 14, Number 5, pages 1164–1175, `https://epubs.siam.org/doi/abs/10.1137/0114093`.

  3. **Fokas Method**

     (a) See Wikipedia entry `https://en.wikipedia.org/wiki/Fokas_method`

     (b) Bernard Deconinck, Thomas Trogdon, and Vishal Vasan, *The Method of Fokas for Solving Linear Partial Differential Equations*, SIAM Review, Volume 56, Number 1, pages 159–186, `https://epubs.siam.org/doi/abs/10.1137/110821871`

4. **Specialized Techniques**
   (a) Barton L. Willis, *An Extensible Differential Equation Solver*, ACM SIGSAM Bulletin, Volume 35, Number 1, March 2001, `http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.577.5712&rep=rep1&type=pdf`
   (b) J F Carinena, J Grabowski, and J de Lucas, *Superposition rules for higher order systems and their applications*, Journal of Physics A: Mathematical and Theoretical Volume 45, Number 18, 20 April 2012, `http://iopscience.iop.org/article/10.1088/1751-8113/45/18/185202/meta`
   (c) Gromov's approach
      – Deane Yang, *Gromov's Approach to Solving Underdetermined Systems of PDE's*, `http://www.deaneyang.com/papers/underdetermined_ode.pdf`
      – Victor Lehenkyi, *The Integrability of Some Underdetermined Systems*, Proceedings of Institute of Mathematics of NAS of Ukraine, 2000, Volume 30, Part 1, 157–164, `https://pdfs.semanticscholar.org/bab0/ebb721ae26a91d517e2ad5277583e6569 pdf`

- Approximate Methods

1. **Renormalization Group Method**
   (a) Hayato Chiba, *Simplified Renormalization Group Method for Ordinary Differential Equations*, `http://www2.math.kyushu-u.ac.jp/~chiba/paper/rg2.pdf`
   (b) R. E. Lee DeVille, Anthony Harkin, Matt Holzer, Kresimir Josic, and Tasso J. Kaper, *Analysis of a Renormalization Group Method for Solving Perturbed Ordinary Differential Equations*, 14 August 2006, `https://faculty.math.illinois.edu/~rdeville/research/FinalRGpaper.pdf`
   (c) Eleftherios Kirkinis, *The Renormalization Group: A Perturbation Method for the Graduate Curriculum*, SIAM REVIEW, Volume 54, Number 2, pages 374–388 `https://epubs.siam.org/doi/pdf/10.1137/080731967`

- Numerical Methods

1. **Block Operators**
   (a) Jared L. Aurentz and Lloyd N. Trefethen, "Block Operators and Spectral Discretizations", *SIAM Review*, Vol. 39, No. 2, pages 423-446, `https://epubs.siam.org/doi/abs/10.1137/16M1065975`
2. **Parallel Computer Algorithms**
   (a) Terry Haut and Beth Wingate, *An Asymptotic Parallel-in-Time Method for Highly Oscillatory PDEs*, SIAM J. Sci. Comput., Volume 36, Number 2, A693–A713, `https://epubs.siam.org/doi/abs/10.1137/130914577`
   (b) Marcos Rodriguez, Fernando Blesa, and Barrio, Roberto, *OpenCL parallel integration of ordinary differential equations: Applications in computational dynamics*, Computer Physics Communications, 192, (July 2015).
3. **Poincare Maps**
   (a) See Wikipedia entry `https://en.wikipedia.org/wiki/Poincare_map`

(b) Warwick Tucker, *Computing accurate Poincaré maps*, Physica D 171 (2002) 127–137, `http://www2.math.uu.se/~warwick/main/papers/accuratePoincare.pdf`

4. **Quantum Algorithms**

   (a) Dominic W. Berry, Andrew M. Childs, Aaron Ostrander, and Guoming Wang, *Quantum algorithm for linear differential equations with exponentially improved dependence on precision*, 17 Feb 2017, `https://arxiv.org/pdf/1701.03684.pdf`

   (b) Dominic W. Berry, *High-order quantum algorithm for solving linear differential equations*, 28 Jan 2014, `https://arxiv.org/pdf/1010.2745.pdf`.

UPDATED MATERIAL

There have been advances in many fields that should be included in an updated version of *Handbook of Differential Equations*. There are also references that were previously overlooked.

- **5. Chaos in Dynamical Systems**, add the references
  - Monica A. Garcia–Nustes and Jorge A. Gonzalez, *Universal functions and exactly solvable chaotic systems*, 7 Nov 2008, `https://arxiv.org/pdf/0811.1179.pdf`.
  - J. C. Sprott, *A simple chaotic delay differential equation*, Physics Letters A 366 (2007), pages 397–402, `http://sprott.physics.wisc.edu/pubs/paper304.pdf`
  - Y. Charles Li, *Chaos in Partial Differential Equations*, 4 September 2009, `https://arxiv.org/pdf/0909.0910.pdf`.
- **6. Classification of Partial Differential Equations**

  Add the reference
  - M. S. Chong, A. E. Perry, and B. J. Cantwell, *A general classification of three-dimensional flow fields*, Physics of Fluids, A 2(5), 1990, pages 765–777.

  Add the new **Example 4**
  - Consider the 3-by-3 system $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$, where A is constant. The eigenvalues of A satisfy $\lambda^3 + P\lambda^2 + Q\lambda + R = 0$ where

    $$\begin{aligned} P &= -\operatorname{trace}(A) \\ Q &= \frac{1}{2}\left(P^2 - \operatorname{trace} A^2\right) \\ R &= -\det A \end{aligned}$$

    The surface S, defined by $27R^2 + \left(4P^3 - 18PQ\right)R + \left(4Q^3 - P^2Q^2\right) = 0$, divides the real solutions from the complex solutions. When $\{P, Q, R\}$ are real, the surface S can be split up into two surfaces S_{1a} and S_{1b} given by $R = R_a$ and $R = R_b$ where:

    $$\begin{aligned} R_a &= \frac{1}{3}P\left(Q - \frac{2}{9}P^2\right) - \frac{2}{27}\left(P^2 - 3Q\right)^{3/2} \\ R_b &= \frac{1}{3}P\left(Q - \frac{2}{9}P^2\right) + \frac{2}{27}\left(P^2 - 3Q\right)^{3/2} \end{aligned}$$

    The classification of solution trajectories in (P, Q, R) space is:

1. node / node / node
   (a) nodes are stable if: $[P > 0,\ 0 < Q < \frac{P^2}{3},\ 0 < R < R_b]$
   (b) nodes are unstable if: $[P < 0,\ 0 < Q < \frac{P^2}{3},\ R_a < R < 0]$
2. node / node / star node
   (a) nodes are stable if: $[P > 0,\ \frac{P^2}{4} < Q < \frac{P^2}{3},\ R = R_a]$ or $[P > 0,\ 0 < Q < \frac{P^2}{3},\ R = R_b]$
   (b) nodes are unstable if: $[P < 0,\ \frac{P^2}{4} < Q < \frac{P^2}{3},\ R = R_a]$ or $[P < 0,\ 0 < Q < \frac{P^2}{3},\ R = R_b]$
3. star node / star node / star node
   (a) nodes are stable if: $[P > 0,\ Q = \frac{P^2}{3},\ R = R_a = R_b = \frac{P^3}{27}]$
   (b) nodes are unstable if: $[P < 0,\ Q = \frac{P^2}{3},\ R = R_a = R_b = \frac{P^3}{27}]$
4. line node-saddle / line node-saddle / node
   (a) nodes are stable if: $[P > 0,\ 0 < Q < \frac{P^2}{4},\ R = 0]$
   (b) nodes are unstable if: $[P < 0,\ 0 < Q < \frac{P^2}{4},\ R = 0]$
5. line node-saddle / line node-saddle / star node
   (a) nodes are stable if: $[P > 0,\ Q = \frac{P^2}{4},\ R = R_a = 0]$
   (b) nodes are unstable if: $[P < 0,\ Q = \frac{P^2}{4},\ R = R_b = 0]$
6. node / saddle / saddle
   (a) node is stable if: $[P \geq 0,\ Q < \frac{P^2}{4},\ R_a < R < 0]$ or $[P < 0,\ Q < 0,\ R_a < R < 0]$
   (b) node is unstable if: $[P \leq 0,\ Q < \frac{P^2}{4},\ 0 < R < R_a]$ or $[P > 0,\ Q < 0,\ 0 < R < R_b]$
7. star node / saddle / saddle
   (a) node is stable if: $[P \geq 0,\ Q < \frac{P^2}{4},\ R = R_a]$ or $[P < 0,\ Q < 0,\ R = R_a]$
   (b) node is unstable if: $[P \leq 0,\ Q < \frac{P^2}{4},\ R = R_b]$ or $[P > 0,\ Q < 0,\ R = R_b]$
8. line node-saddle / line node-saddle / no flow
   (a) both line node-saddles are stable if: $[P > 0,\ Q = 0,\ R = 0]$
   (b) both line node-saddles are unstable if: $[P < 0,\ Q = 0,\ R = 0]$
   (c) one line node-saddle is stable and the other unstable if: [for all P, $Q < 0$, $R = 0$]
9. focus / stretching
   (a) focus is stable if: $[P \geq 0,\ Q > \frac{P^2}{4},\ R < 0]$ or $[P \geq 0,\ Q < \frac{P^2}{4},\ R < R_a]$ or $[P < 0,\ Q > 0,\ R < PQ]$ or $[P < 0,\ Q > 0,\ R < R_a]$
   (b) focus is unstable if: $[P < 0,\ Q > \frac{P^2}{3},\ PQ < R < 0]$ or $[P < 0,\ \frac{P^2}{4} < Q < \frac{P^2}{3},\ R_b < R < 0]$ or $[P < 0,\ 0 < Q < \frac{P^2}{3},\ PQ < R < R_a]$
10. focus / compressing
    (a) focus is stable if: $[P > 0,\ Q > \frac{P^2}{4},\ 0 < R < PQ]$ or $[P > 0,\ \frac{P^2}{4} < Q < \frac{P^2}{3},\ 0 < R < R_a]$ or $[P > 0,\ 0 < Q < \frac{P^2}{3},\ R_b < R < PQ]$
    (b) focus is unstable if: $[P \leq 0,\ Q > \frac{P^2}{4},\ R > 0]$ or $[P \leq 0,\ Q < \frac{P^2}{4},\ R < R_b]$ or $[P > 0,\ Q > 0,\ R < PQ]$ or $[P > 0,\ Q > 0,\ R < R_a]$
11. focus / no flow
    (a) focus is stable if: $[P > 0,\ Q > \frac{P^2}{4},\ R = 0]$
    (b) focus is unstable if: $[P < 0,\ Q > \frac{P^2}{4},\ R = 0]$
12. center
    (a) stretching if $[P < 0,\ Q > 0,\ R = PQ]$
    (b) compressing if $[P > 0,\ Q > 0,\ R = PQ]$

(c) no flow if $[P = 0,\ Q > 0,\ R = 0]$

- **8. Conservation Laws**, add the references
  - J. J. H. Bashingwa and A. H. Kara, *A basis of hierarchy of generalized symmetries and their conservation laws for the (3+1)-dimensional diffusion equation*, 6 Dec 2017, `https://arxiv.org/pdf/1712.02117.pdf`.
  - Wen-Xiu Ma, *Conservation laws by symmetries and adjoint symmetries*, 11 June 2017, `https://arxiv.org/pdf/1707.03496.pdf`.
  - Roman O. Popovych and Alexander Bihlo, *Inverse problem on conservation laws*, 9 May 2017, `https://arxiv.org/pdf/1705.03547.pdf`.
- **84. Pfaffian Differential Equations**, add the reference
  - David Delphenich, *The role of integrability in a large class of physical system*, 16 March 2017, `https://arxiv.org/ftp/arxiv/papers/1210/1210.4976.pdf`.
- **66. Factoring Operators**

  Add the reference
  - Mark Giesbrecht, Albert Heinle, and Viktor Levandovskyy, *Factoring Differential Operators in n Variables*, 31 March 2014, `https://arxiv.org/pdf/1404.0002.pdf`.

  Add the following

  The paper by Giesbrecht *et al.*, presents an algorithm for factoring partial differential operators for a large class of operators. Technically they find factorizations over many algebras, including the polynomial n^{th} Weyl algebra. The n^{th} Weyl algebra Q_n is defined by

$$Q_n := \mathbb{K}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle$$
$$\partial_i x_j = \begin{cases} x_j \partial_i & \text{if } i \neq j \\ q_i x_j \partial_i + 1 & \text{if } i = j \end{cases}$$
$$0 = \partial_i \partial_j - \partial_j \partial_i = x_i x_j - x_j x_i$$

  for $i, j = 1, \ldots, n$ where the $\{q_1, \ldots, q_n\}$ are units in $\mathbb{K}$. A computer algebra implementation of their algorithm is available in Singular.

  Examples in the paper include:
  1. $\partial^2 = \partial\partial = \left(\partial + \frac{1}{x+c}\right)\left(\partial - \frac{1}{x+c}\right)$
  2. $\partial^3 - x\partial - 2 = \left(\partial + \frac{1}{x}\right)\left(\partial^2 - \frac{1}{x}\partial - x\right)$
  3. $x\left(\partial^3 - x\partial - 2\right) = \partial\left(x\partial^2 - x^2 - \partial\right)$
  4. $\left(\partial_1 + 1\right)^2\left(\partial_1 + x_1\partial_2\right) = \left(x_1\partial_1\partial_2 + \partial_1^2 + x_1\partial_2 + 2\partial_2\right)\left(\partial_2 + 1\right).$

  Additionally, using their package,
  - 60 distinct factorizations are found for $x_1 x_2^2 x_3^2 \partial_1 \partial_2^2 + x_2 x_3^3 \partial_2$
  - 60 distinct factorizations are found for $\left(x_1^2\partial_1 + x_1 x_2 \partial_2\right)\left(\partial_1\partial_2 + \partial_1^2\partial_2^2 x_1 x_2\right)$

– 12 distinct factorizations are found for $\left(x_1^4-1\right)x_1\partial_1^2+\left(1+7x_1^4\right)\partial_1+8x_1^3$.

- **95. Variation of parameters**

  Add the following as note number 5:

  Explicit solutions using variation of parameters:

  1. If the linear second order equation $L[y]=y''+P_1(x)y'+P_0(x)y=R(x)$ has the homogeneous solutions $y_1(x)$ and $y_2(x)$ (i.e., $L[y_i]=0$), then the solution to the original equation may be written as

  $$\begin{aligned} y(x) &= -y_1(x)\int \frac{y_2(x)R(x)}{W(y_1,y_2)}\,dx + y_2(x)\int \frac{y_1(x)R(x)}{W(y_1,y_2)}\,dx,\\ &= y_1(x)\int \frac{\begin{vmatrix} 0 & y_2\\ R & y_2'\end{vmatrix}}{W(y_1,y_2)}\,dx + y_2(x)\int \frac{\begin{vmatrix} y_1 & 0\\ y_1'' & R\end{vmatrix}}{W(y_1,y_2)}\,dx \end{aligned}$$

  where $W(y_1,y_2)=y_1y_2'-y_1'y_2=\begin{vmatrix} y_1 & y_2\\ y_1' & y_2'\end{vmatrix}$ is the Wronskian.

  2. If the linear third order equation $L[y]=y'''+P_2(x)y''+P_1(x)y'+P_0(x)y=R(x)$ has the homogeneous solutions $y_1(x)$, $y_2(x)$, and $y_3(x)$ (i.e., $L[y_i]=0$), then the solution to the original equation may be written as

  $$\begin{aligned} y(x) = y_1(x)\int \frac{\begin{vmatrix} 0 & y_2 & y_3\\ 0 & y_2' & y_3'\\ R & y_2'' & y_3''\end{vmatrix}}{W(y_1,y_2,y_3)}\,dx &+ y_2(x)\int \frac{\begin{vmatrix} y_1 & 0 & y_3\\ y_1' & 0 & y_3'\\ y_1'' & R & y_3''\end{vmatrix}}{W(y_1,y_2,y_3)}\,dx\\ &+ y_3(x)\int \frac{\begin{vmatrix} y_1 & y_2 & 0\\ y_1' & y_2' & 0\\ y_1'' & y_2'' & R\end{vmatrix}}{W(y_1,y_2,y_3)}\,dx \end{aligned} \tag{5}$$

  where $W(y_1,y_2,y_3)=\begin{vmatrix} y_1 & y_2 & y_3\\ y_1' & y_2' & y_3'\\ y_1'' & y_2'' & y_3''\end{vmatrix}$ is the Wronskian.

- **112. Separation of Variables**, add the references

  – H. S. Cohl and H. Volkmer, *Separation of variables in an asymmetric cyclidic coordinate system*, 16 Jan 2013, `https://arxiv.org/pdf/1301.3559.pdf`.

  – H. S. Cohl and H. Volkmer, *Expansions for a fundamental solutionl of Laplace's equation on R^3 in 5-cyclidic harmonics*, 14 Nov 2013, `https://arxiv.org/pdf/1311.3514.pdf`.

  – George Pogosyan, Alexey Sissakian, and Pavel Winternitz, *Separation of Variables and Lie Algebra Contractions. Applications to Special Functions*, 8 Oct 2003, `https://arxiv.org/pdf/math-ph/0310011.pdf`.

  – A. Szereszewski and A. Sym, *On Darboux's approach to R-separability of variables. Classification of conformally flat 4-dimensional binary metrics*, Journal of Physics A: Mathematical and Theoretical, Volume 48, Number 38, 25 August 2015 `http://iopscience.iop.org/article/10.1088/1751-8113/48/38/385201/meta`.

- **128 Interval Analysis**, add the reference
  - Fahimeh Goodarzi, Mahmoud Hadizadeh, and Farideh Ghoreishi, *An interval solution for the n-th order linear ODEs with interval initial conditions*, Mathematical Communications, Volume 18, Number 1, 2013, http://www.mathos.unios.hr/mc/index.php/mc/article/view/242.
- **156. Available Software**, add the references
  - FreeFem++ – a numerical PDE solver – http://www.freefem.org/
  - JuliaDiffEq – a numerical differential equations solver for Julia – http://juliadiffeq.org/packages.html
  - deSolve – a numerical ODE/ADE/DDE solver in R – http://cran.ms.unimelb.edu.au/web/packages/deSolve/vignettes/deSolve.pdf – https://cran.r-project.org/web/packages/deSolve/index.html
  - SUNDIALS: SUite of Nonlinear and DIfferential/ALgebraic Equation Solvers – https://computation.llnl.gov/projects/sundials
- **182. Integrating Stochastic Equations**, add the references
  - Desmond J. Higham, *An Algorithmic Introduction to Numerical Simulation of Stochastic Differential Equations*, SIAM REVIEW, Volume 43, Number 3, 2001, pages 525–546, https://epubs.siam.org/doi/pdf/10.1137/S0036144500378302
  - Cian O. Mahony, *The Numerical Analysis of Stochastic Differential Equations*, 8 September 2006, http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.117.8043&rep=rep1&type=pdf
  - Kevin Burrage and Pamela Burrage, *High Strong Order Explicit Runge-Kutta Methods for Stochastic Ordinary Differential Equations*, Applied Numerical Mathematics, 22 (1–3), October 1996, https://www.sciencedirect.com/science/article/pii/S016892749600027X